Homework 5 due Thursday, October 9.

Pages 58–60, problems 4.2, 4.7, 4.12, 4.17, 4.20, 4.24, 4.26.

Assume all rings are commutative with $0 \neq 1$.

4.2: Let $R = \{r_1, r_2, \ldots, r_n\}$ where $r_1 = 1$. Given $r_i \in R$ such that $r_i \neq 0$, explain why $(r_i r_1, r_i r_2, \ldots, r_i r_n)$ is a permutation of $(r_1, r_2, \ldots r_n)$ and why this implies that $r_i^{-1} \in R$.

4.7: (a) You must prove that $x, y \in N$ and $r \in R$ implies $x + y \in N$ and $rx \in N$. (b) You must prove that $(N + r)^n = N + 0$ implies $N + r = N + 0$.

4.12: By definition, $I + J = \{i + j : i \in I, j \in J\}$. You must prove that $x, y \in I + J$ and $r \in R$ implies $x + y \in I + J$ and $rx \in I + J$.

4.17: Under the assumption that K is an ideal of R, you must prove that $x, y \in J$ and $r \in R$ implies $rx \in R$.

4.20: We proved in class that if $\phi : R \to S$ is a ring epimorphism then $\ker(\phi)$ is an ideal of R and $R/\ker(\phi) \cong S$. So if S is an integral domain then so is $R/\ker(\phi)$, which makes $\ker(\phi)$ into a prime ideal of R. So all you have to do is choose an integral domain S which is not a field and a ring epimorphism $\phi : \mathbb{Z} \times \mathbb{Z} \to S$ and prove that ϕ has the desired properties and calculate $\ker(\phi)$.

4.24: For (d), if you can show that S/I is an integral domain then by Exercise [4.2] it must be a field, which makes I a maximal ideal of S.

4.26: An example of this would be the ring $\mathbb{Z}[\sqrt{-6}] = \{a + b\sqrt{-6} : a, b \in \mathbb{Z}\}$. For (a) you must verify that $x, y \in \mathbb{Z}[\alpha]$ implies $x - y \in \mathbb{Z}[\alpha]$ and $xy \in \mathbb{Z}[\alpha]$. For (b), an automorphism of $\mathbb{Z}[\alpha]$ is a ring homomorphism $\phi : \mathbb{Z}[\alpha] \to \mathbb{Z}[\alpha]$ which is bijective. Assuming that ϕ is an arbitrary automorphism, prove that $\phi(1) = 1$, then prove that $\phi(k) = k$ for all $k \in \mathbb{Z}$, then prove that $\phi(\alpha)^2 = a$, then find all possible formulas for $\phi(c + d\alpha)$ where $c, d \in \mathbb{Z}$.