Exam 4 Solutions Math 641 Fall 2014

Let $K = \mathbb{Q}$, $u = \sqrt{4+\sqrt{8}}$, $\tau = K[u]$.

(a) Compute $[\tau : K]$, proving carefully that your $\operatorname{irr}(u, K)$ is actually irreducible.

(b) Prove that τ is a splitting field of $\operatorname{irr}(u, K)$. You should be able to express each root in terms of u.

(c) By a homework exercise, each $\tau \in \operatorname{Gal}_K(F)$ is determined by $\tau(u)$. Compute $\tau_i \circ \tau_j(u)$ for each pair $\tau_i, \tau_j \in \operatorname{Gal}_K(F)$, using the information in (b). Use this information to construct the group multiplication table for $\operatorname{Gal}_K(F)$ and to find an isomorphism between $\operatorname{Gal}_K(F)$ and one of the standard groups we have studied in this course. This will be useful for finding all the subgroups of $\operatorname{Gal}_K(F)$ in Part (d).

(d) Compute H' for each subgroup H of $\operatorname{Gal}_K(F)$. Arrange the subgroups and the intermediate fields into two Hasse diagrams. See pp. 132–133 for an example of Hasse diagrams and the process of finding fixed fields. There is also a Mathematica notebook on the course webpage which has examples of fixed fields.

(e) The field extension $K \subseteq F$ meets the hypotheses of the Fundamental Theorem of Galois Theory since it is a splitting field of characteristic zero. This theorem guarantees that when H is a normal subgroup of $\operatorname{Gal}_K(F)$, H' is normal over K, which implies that H' is a splitting field of some polynomial $f_H(x)$ in $K[x]$. Given each normal subgroup H, find a suitable polynomial $f_H(x)$.

Solutions:

(a)

$$u^2 = 4 + \sqrt{8}$$

$$(u^2 - 4)^2 = 8$$

$$u^4 - 8u^2 + 8 = 0$$

$x^4 - 8x^2 + 8$ is irreducible in $\mathbb{Q}[x]$: The rational roots of $x^4 - 8x^2 + 8$ belong to $\{\pm 1, \pm 2, \pm 4, \pm 8\}$, none of which are actual roots. Hence $x^4 - 8x^2 + 8$ has no linear factors in $\mathbb{Q}[x]$. If $x^4 - 8x^2 + 8$ has a quadratic factor in $\mathbb{Q}[x]$ then $x^4 - 8x + 8 = (x^2 + ax + b\epsilon)(x^2 + cx + d\epsilon)$ where $|\epsilon| = 1$, $a, c \in \mathbb{Z}$,

$(b,d) \in \{(1,8),(2,4)\}$. Comparing coefficients of x^3, x^2, and x, $a+c=0$, $ac+b\epsilon+d\epsilon=-8$, $bc\epsilon+ad\epsilon=0$. Hence $bc+ad=0$, $c(b-d)=0$, $c=0$, $\epsilon=-1$, $b+d=8$, a contradiction. So x^4-8x^2+8 is irreducible in $\mathbb{Q}[x]$. Hence $\operatorname{irr}(u,\mathbb{Q})=x^4-8x^2+8$.

Another solution: If x^4-8x^2+8 is reducible in $\mathbb{Q}[x]$ then it is reducible in $\mathbb{Z}[x]$, hence in $\mathbb{Z}_3[x]$, hence it must be possible to divide x^4-8x^2+8 by a monic polynomial of degree 1 or 2 in $\mathbb{Z}_3[x]$ and obtain a remainder of 0. There are 3 monic polynomials of degree 1 in $\mathbb{Z}_3[x]$ and 9 monic polynomials of degree 2 in $\mathbb{Z}_3[x]$, and dividing x^4-8x^2+8 by each of these 12 polynomials in $\mathbb{Z}_3[x]$ leaves a non-zero remainder in every case. Contradiction. Therefore x^4-8x^2+8 is irreducible in $\mathbb{Q}[x]$.

(b) The roots of x^4-8x^2+8 in $\mathbb{C}$ are

$$(u_1,u_2,u_3,u_4)=(\sqrt{4+\sqrt{8}},-\sqrt{4+\sqrt{8}},\sqrt{4-\sqrt{8}},-\sqrt{4-\sqrt{8}}).$$

We must show that each root belongs to $\mathbb{Q}[u]$ where $u=\sqrt{4+\sqrt{8}}$. We have $u_1=u$ and $u_2=-u$. We also have

$$\frac{1}{u}=\frac{1}{\sqrt{4+\sqrt{8}}}=\frac{\sqrt{4-\sqrt{8}}}{\sqrt{8}}=\frac{u_3}{u^2-4},$$

hence $u_3=\frac{u^2-4}{u}$ and $u_4=-\frac{u^2-4}{u}$. Therefore $u_1,u_2,u_3,u_4\in\mathbb{Q}[u]$.

(c) By the Galois Group Algorithm, v_1 can be any root of x^4-8x^2+8 in $\mathbb{Q}[u]$. This yields $\operatorname{Gal}_K(F)=\{\tau_1,\tau_2,\tau_3,\tau_4\}$ where

$$\tau_1(u)=\sqrt{4+\sqrt{8}}=u,$$

$$\tau_2(u)=-\sqrt{4+\sqrt{8}}=-u,$$

$$\tau_3(u)=\sqrt{4-\sqrt{8}}=\frac{u^2-4}{u},$$

$$\tau_4(u)=-\sqrt{4-\sqrt{8}}=-\frac{u^2-4}{u}.$$

Compositions: we have, for example, $\tau_3(\tau_4(u))=\tau_3(-\frac{u^2-4}{u})=-\frac{\tau_3(u)^2-4}{\tau_3(u)}=-\frac{-\sqrt{8}}{\sqrt{4-\sqrt{8}}}=\sqrt{4+\sqrt{8}}=u$, which implies $\tau_3\circ\tau_4=\tau_1$. Below we work out the multiplication tables for both $\text{Gal}_K(F)$ (under function composition) and $\mathbb{Z}_4$ (under addition mod 4). We can see that $\text{Gal}_K(F) \cong \mathbb{Z}_4$ via the isomorphism ϕ defined by $\phi(\tau_1) = 0$, $\phi(\tau_2) = 2$, $\phi(\tau_3) = 1$, $\phi(\tau_4) = 3$. We also have $\text{Gal}_K(F) = \langle \tau_3 \rangle$.

$\circ$	τ_1	τ_2	τ_3	τ_4
τ_1	τ_1	τ_2	τ_3	τ_4
τ_2	τ_2	τ_1	τ_4	τ_3
τ_3	τ_3	τ_4	τ_2	τ_1
τ_4	τ_4	τ_3	τ_1	τ_2

+	0	2	1	3
0	0	2	1	3
2	2	0	3	1
1	1	3	2	0
3	3	1	0	2

(d) The subgroups of $\mathbb{Z}_4$ are

$$H_1 = \langle 0 \rangle = \{0\},$$

$$H_2 = \langle 2 \rangle = \{0, 2\},$$

$$H_3 = \langle 1 \rangle = \{0, 1, 2, 3\}.$$

Hence the subgroups of $\text{Gal}_K(F)$ are

$$H_1 = \langle \tau_1 \rangle = \{\tau_1\},$$

$$H_2 = \langle \tau_2 \rangle = \{\tau_1, \tau_2\},$$

$$H_3 = \langle \tau_3 \rangle = \{\tau_1, \tau_3, \tau_2, \tau_4\}.$$

We have

$$H_1' = \{x \in F : \tau_1(x) = x\} = F = \mathbb{Q}[u]$$

$$H_2' = \{x \in F : \tau_1(x) = \tau_2(x) = x\} = \text{to be determined},$$

$$H_3' = \{x \in F : \tau_1(x) = \tau_3(x) = \tau_2(x) = \tau_4(x) = x\} =$$

$$(\text{Gal}_K(F))' = (K')' = K'' = K = \mathbb{Q}.$$

A basis for F over $\mathbb{Q}$ is $\{1, u, u^2, u^3\}$ and the typical element in F is $a + bu + cu^2 + du^3$ where $a, b, c, d \in \mathbb{Q}$. This element belongs to H_2' iff it is fixed by τ_2. In other words,

$$a + bu + cu^2 + du^3 \in H_2' \Leftrightarrow$$

$$\tau_2(a + bu + cu^2 + du^3) = a + bu + cu^2 + du^3 \Leftrightarrow$$

$$a - bu + cu^2 - du^3 = a + bu + cu^2 + du^3 \Leftrightarrow$$

$$b = d = 0.$$

Hence

$$H_2' = \{a + cu^2 : a, c \in \mathbb{Q}\} = \mathbb{Q}[u^2] = \mathbb{Q}[4 + \sqrt{8}] = \mathbb{Q}[\sqrt{8}].$$

So the Hasse diagrams are

$$\langle \tau_1 \rangle \subseteq \langle \tau_2 \rangle \subseteq \langle \tau_3 \rangle$$

and

$$\mathbb{Q}[\sqrt{4 + \sqrt{8}}] \supseteq \mathbb{Q}[\sqrt{8}] \supseteq \mathbb{Q}.$$

(d) Since $\mathrm{Gal}_K(F)$ is abelian, every subgroup is normal. $H_1' = \mathbb{Q}[\sqrt{4 + \sqrt{8}}]$ is a splitting field of $x^4 - 8x^2 + 8$, $H_2' = \mathbb{Q}[\sqrt{8}]$ is a splitting field of $x^2 - 8$, and $H_3' = \mathbb{Q}$ is a splitting field of $x - 1$.