Exam 3 Solutions Math 641 Fall 2014

Let $u_1, u_2, u_3, u_4 \in \mathbb{C}$ be the roots of $x^4 + 2x^2 - 7$. Let $E = \mathbb{Q}[u_1, u_2, u_3, u_4]$ and $G = \mathrm{Gal}_{\mathbb{Q}}(E)$.

(a) Using the Galois Group Algorithm, compute the elements of G. Make sure to prove that your irreducible polynomials are actually irreducible.

(b) Is G isomorphic to $\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, D_4, or none of these?

(c) Compute $f(\sqrt{7/8}\, i)$ for each $f \in G$..

Solution: The roots of $x^4 + 2x^2 - 7$ are $\pm\sqrt{-1 \pm \sqrt{8}}$. We will set

$$u_1 = \sqrt{-1+\sqrt{8}}$$

$$u_2 = -\sqrt{-1+\sqrt{8}}$$

$$u_3 = \sqrt{-1-\sqrt{8}}$$

$$u_4 = -\sqrt{-1-\sqrt{8}}.$$

Setting

$$U_1 = \sqrt{-1+\sqrt{8}}$$

$$U_2 = \sqrt{-1-\sqrt{8}}$$

we have

$$\mathbb{Q}[u_1, u_2, u_3, u_4] = \mathbb{Q}[U_1, U_2].$$

Claim: $\mathrm{irr}(U_1, \mathbb{Q}) = x^4 + 2x^2 - 7$. To prove this, suppose $x^4 + 2x^2 - 7 = a(x)b(x)$ where $a(x), b(x) \in \mathbb{Q}[x]$. Then $x^4 + 2x^2 - 7 = A(x)B(x)$ where $A(x), B(x) \in \mathbb{Z}[x]$ are monic polynomials. The only possible linear factors are therefore $x+7$, $x+1$, $x-1$, and $x-7$, but one can check that none of the numbers in the set $\{-7, -1, 1, 7\}$ are roots of x^4+2x^2-7. If $A(x)$ and $B(x)$ are quadratic then we must have $x^4 + 2x^2 - 7 = (x^2 + hx + \epsilon)(x^2 + kx - 7\epsilon)$ where $|\epsilon| = 1$. Expanding this,

$$x^4 + 2x^2 - 7 = x^4 + (h+k)x^3 + (hk - 6\epsilon)x^2 + \epsilon(k - 7h)x - 7.$$

Comparing coefficients of x^3 and x, we must have $h+k=0$ and $k-7h=0$. This forces $h=k=0$, which implies

$$x^4+2x^2-7=x^4-6\epsilon x^2-7.$$

This is impossible because $|\epsilon|=1$. So the claim is proved.

Claim: $\text{irr}(U_2,\mathbb{Q}[U_1])=x^2+U_1^2+2$. It's clear that $U_2^2+U_1^2+2=0$. Moreover, since U_2 is pure imaginary while U_1 is real, $U_2\notin\mathbb{Q}[U_1]$, so U_2 is not the root of a monic linear polynomial in $\mathbb{Q}[U_1][x]$.

Constructing the Galois Group: v_1 can be any root of x^4+2x^2-7, hence $v_1\in\{U_1,-U_1,U_2,-U_2\}$.

Case 1: $v_1=U_1$. This implies $\text{irr}(U_2,\mathbb{Q}[U_1])'=x^2+U_1^2+2=x^2+1+\sqrt{8}$ and $v_2=\pm\sqrt{-1-\sqrt{8}}=\pm U_2$.

Case 2: $v_1=-U_1$. This implies $\text{irr}(U_2,\mathbb{Q}[U_1])'=x^2+U_1^2+2=x^2+1+\sqrt{8}$ and $v_2=\pm\sqrt{-1-\sqrt{8}}=\pm U_2$.

Case 3: $v_1=U_2$. This implies $\text{irr}(U_2,\mathbb{Q}[U_1])'=x^2+U_2^2+2=x^2+1-\sqrt{8}$ and $v_2=\pm\sqrt{-1+\sqrt{8}}=\pm U_1$.

Case 4: $v_1=-U_2$. This implies $\text{irr}(U_2,\mathbb{Q}[U_1])'=x^2+U_2^2+2=x^2+1-\sqrt{8}$ and $v_2=\pm\sqrt{-1+\sqrt{8}}=\pm U_1$.

We have identified 8 Galois Group elements:

$$f_1: a(U_1,U_2)\mapsto a(U_1,U_2)$$
$$f_2: a(U_1,U_2)\mapsto a(U_1,-U_2)$$
$$f_3: a(U_1,U_2)\mapsto a(-U_1,U_2)$$
$$f_4: a(U_1,U_2)\mapsto a(-U_1,-U_2)$$
$$f_5: a(U_1,U_2)\mapsto a(U_2,U_1)$$
$$f_6: a(U_1,U_2)\mapsto a(U_2,-U_1)$$
$$f_7: a(U_1,U_2)\mapsto a(-U_2,U_1)$$
$$f_8: a(U_1,U_2)\mapsto a(-U_2,-U_1).$$

It is not difficult to check that $f_6 \circ f_5 = f_3$ and that $f_5 \circ f_6 = f_2$, hence G is not abelian. Setting $S = f_6$ and $T = f_5$, one can check that $o(S) = 4$, $o(T) = 2$, and $TS = S^3 T$. Hence $G \cong D_4$.

To compute $f(\sqrt{7/8}\ i)$ for each $f \in G$ we must first express $\sqrt{7/8}\ i$ in terms of U_1 and U_2. We have:

$$\begin{aligned}
U_1 &= \sqrt{-1+\sqrt{8}} \\
U_2 &= \sqrt{1+\sqrt{8}}\ i \\
U_1 U_2 &= \sqrt{7}\ i \\
U_1^2 + 1 &= \sqrt{8} \\
U_2^2 + 1 &= -\sqrt{8} \\
\frac{U_1 U_2}{U_1^2+1} &= \sqrt{7/8}\ i
\end{aligned}$$

Since each $f \in G$ is a field isomorphism,

$$f(\sqrt{7/8}\ i) = f\left(\frac{U_1 U_2}{U_1^2+1}\right) = \frac{f(U_1)f(U_2)}{f(U_1)^2+1}.$$

This implies

$$\begin{aligned}
f_1(\sqrt{7/8}\ i) &= \frac{U_1U_2}{U_1^2+1} = \sqrt{7/8}\ i \\
f_2(\sqrt{7/8}\ i) &= \frac{-U_1U_2}{U_1^2+1} = -\sqrt{7/8}\ i \\
f_3(\sqrt{7/8}\ i) &= \frac{-U_1U_2}{U_1^2+1} = -\sqrt{7/8}\ i \\
f_4(\sqrt{7/8}\ i) &= \frac{U_1U_2}{U_1^2+1} = \sqrt{7/8}\ i \\
f_5(\sqrt{7/8}\ i) &= \frac{U_2U_1}{U_2^2+1} = -\sqrt{7/8}\ i \\
f_6(\sqrt{7/8}\ i) &= \frac{-U_2U_1}{U_2^2+1} = \sqrt{7/8}\ i \\
f_7(\sqrt{7/8}\ i) &= \frac{-U_2U_1}{U_2^2+1} = \sqrt{7/8}\ i \\
f_8(\sqrt{7/8}\ i) &= \frac{U_2U_1}{U_2^2+1} = -\sqrt{7/8}\ i
\end{aligned}$$