Proof of the Cauchy-Binet Theorem and the Matrix Tree Theorem

Cauchy-Binet Theorem: Assume $p \leq q$. Let $A = (a_{ij})$ be an $p \times q$ matrix, let $B = (b_{ij})$ be a $q \times p$ matrix, and write $AB = C = (c_{ij})$. Then

$$\det(AB) = \det(C_1, \ldots, C_p) = \det(\sum_{i=1}^{q} b_{i1} A_i, \ldots, \sum_{i=1}^{q} b_{ip} A_i) =$$

$$\sum_{1 \leq i_1, \ldots, i_p \leq q} b_{i_1 1} \cdots b_{i_p p} \det(A_{i_1}, \ldots, A_{i_p}) =$$

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq q} \sum_{\sigma \in \mathcal{S}_p} b_{i_{\sigma(1)} 1} \cdots b_{i_{\sigma(p)} p} \det(A_{i_{\sigma(1)}}, \ldots, A_{i_{\sigma(p)}}) =$$

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq q} \sum_{\sigma \in \mathcal{S}_p} b_{i_{\sigma(1)} 1} \cdots b_{i_{\sigma(p)} p} \operatorname{sgn}(\sigma) \det(A_{i_1}, \ldots, A_{i_p}) =$$

$$\sum_{I \in \binom{[q]}{p}} \det(A_I) \det(B_I)$$

where for a subset I of $[q]$ of size p, A_I is the submatrix of A using the p columns from I and B_I is the submatrix of B using the p rows from I.

Counting spanning trees: Let G be a graph with vertex set $V = \{x_1, \ldots, x_n\}$ and edge set $E = \{e_1, \ldots, e_m\}$ where $m \geq n - 1$. Then the number of spanning trees of G is

$$\sum_{H \in \binom{E}{n-1}} \chi(H \text{ is a spanning tree}).$$

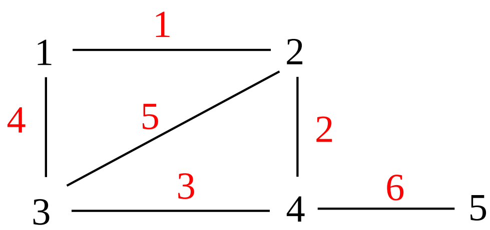
Given the resemblance of this formula to the Cauchy-Binet Theorem, it should not be surprising that there is a determinant formula for this expression.

Matrix-Tree Theorem: Let

$$C = ((-1)^{\chi(x_i = \min e_j)} \chi(x_i \in e_j))$$

where $1 \leq i \leq n - 1$ and $1 \leq j \leq m$. Then the number of spanning trees is $\det(CC^T)$.

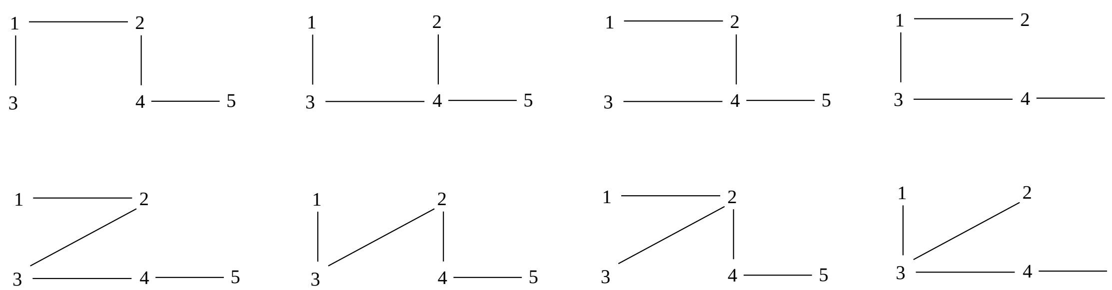
Example:

$G =$ 

$$C = \begin{bmatrix} -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \end{bmatrix}$$

$$\det(CC^T) = 8$$

Spanning trees:



Proof of the Matrix-Tree Theorem: We have

$$\det(CC^T) = \sum_{I \in \binom{[m]}{n-1}} \det(C_I)\det(C_I^T) = \sum_{I \in \binom{[m]}{n-1}} \det(C_I)^2.$$

We will prove that

$$\det(C_I)^2 = \chi(\{e_i : i \in I\} \text{ is a spanning tree})$$

for each $I \in \binom{[m]}{n-1}$.

Let $I \in \binom{[m]}{n-1}$ be given. Name the corresponding edges $f_1, \ldots, f_{n-1}$. Then the ij-entry of C_I is 0 if $x_i \notin f_j$ and is ± 1 if $x_i \in f_j$. These edges form a spanning tree if and only if they are connected and encompass n vertices.

Case 1: $\{f_1, \ldots, f_{n-1}\}$ does not incorporate all n vertices. If x_n is isolated then each column of C_I has a 1 and a -1 in it, so the sum of its columns is the 0 vector, so its columns are linearly dependent and $\det(C_I) = 0$. If some other vertex x_k is isolated then row k in C_I is the 0 vector, which again implies $\det(C_I) = 0$.

Case 2: $\{f_1, \ldots, f_{n-1}\}$ encompasses all n vertices but is not connected. Each component has at least two vertices. The sum of all the rows corresponding to vertices in a component not containing n is 0, hence the rows are not linearly independent and $\det(C_I) = 0$.

Case 3: $\{f_1, \ldots, f_{n-1}\}$ incorporates all n vertices and is connected. The collection of edges forms a spanning tree. Clipping leaf vertices and edges, we can permute the rows and columns of C_I to produce a lower-triangular matrix with ± 1 in each diagonal entry. This implies $\det(C_I) = \pm 1$.