Graphing and constrained optimization using properties of real symmetric matrices

1. Graph $ax^2 + bxy + cy^2 = 1$. In matrix form, this reads

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}.$$

We have already proved that for each symmetric matrix A there is a rotation matrix C of eigenvectors of A such that $A = CDC^T$ where D is a diagonal matrix. Making the substitution we obtain

$$\begin{bmatrix} x & y \end{bmatrix} C \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} C^T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}.$$

Writing

$$\begin{bmatrix} X \\ Y \end{bmatrix} = C^T \begin{bmatrix} x \\ y \end{bmatrix}$$

we obtain

$$\begin{bmatrix} X & Y \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}.$$

In other words,

$$\lambda_1 X^2 + \lambda_2 Y^2 = 1.$$

This is much easier to graph. Since

$$\begin{bmatrix} x \\ y \end{bmatrix} = C \begin{bmatrix} X \\ Y \end{bmatrix}$$

and C is a rotation matrix, all we have to do is identify the angle of rotation θ and rotate the XY graph by θ to obtain the xy graph.

Example: Graph $x^2 + 3xy + 5y^2 = 1$. The eigenvalues of $\begin{bmatrix} 1 & 3/2 \\ 3/2 & 5 \end{bmatrix}$ are $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = \frac{11}{2}$. Eigenspace bases are $\{\begin{bmatrix} 3 \\ -1 \end{bmatrix}\}$ and $\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\}$. This yields

$$C = \begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ -1/\sqrt{10} & 3\sqrt{10} \end{bmatrix}.$$

Identifying this with

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

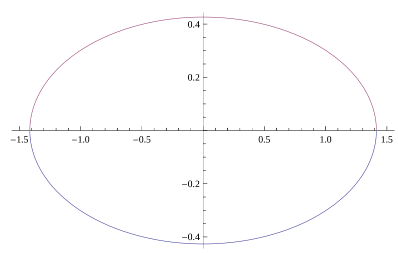
yields $\cos\theta = \frac{3}{\sqrt{10}}$, $\sin\theta = -\frac{1}{\sqrt{10}}$, $\tan\theta = -\frac{1}{3}$, $\theta = \tan^{-1}\frac{-1}{3} = -0.321751$ radians or -18.4349 degrees. So we graph $\frac{1}{2}X^2 + \frac{11}{2}Y^2 = 1$, then rotate -18.4349 degrees. For example, one solution to $\frac{1}{2}X^2 + \frac{11}{2}Y^2 = 1$ is $X = \sqrt{2}$, $Y = 0$. This yields solution

$$\begin{bmatrix} x \\ y \end{bmatrix} = C\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ -1/\sqrt{10} & 3\sqrt{10} \end{bmatrix}\begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{bmatrix}.$$

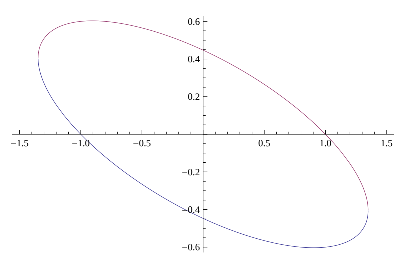
We have

$$(3/\sqrt{5})^2 + 3(3/\sqrt{5})(-1/\sqrt{5}) + 5(-1/\sqrt{5})^2 = 1.$$

Graph of $\frac{1}{2}X^2 + \frac{11}{2}Y^2 = 1$:



Graph of $x^2 + 3xy + 5y^2 = 1$:



2. Find the maximum and minimum value of $x^2 + 3xy + 5y^2$ subject to $x^2 + y^2 = 1$. Given that (x, y) is related to (X, Y) by a rotation, $x^2 + y^2 = 1$

is equivalent to $X^2 + Y^2 = 1$. So equivalently we can find the maximum of $\frac{1}{2}X^2 + \frac{11}{2}Y^2$ subject to $X^2 + Y^2 = 1$. Writing $Y^2 = 1 - X^2$ we want the maximum of $\frac{11}{2} - 5X^2$ where $|X| \leq 1$. The maximum is $\frac{11}{2}$ using $(X, Y) = (0, 1)$, $(x, y) = (1/\sqrt{10}, 3/\sqrt{10})$. The minimum is $\frac{1}{2}$ using $(X, Y) = (1, 0)$, $(x, y) = (3/\sqrt{10}, -1/\sqrt{10})$.