Final Exam, Math 641, Fall 2012

1. Let $K \subseteq F$ be a field extension and let $u \in F$. Let L be the intersection of all subfields L of F such that $K \subseteq L$ and $u \in L$. Let

$$M = \{f(u)g(u)^{-1} \in F : f(x) \in K[x],\ g(x) \in K[x],\ g(u) \neq 0\}.$$

Prove that $L = M$. Hint: Prove that M is a field that contains both K and u. Therefore $L \subseteq M$. Explain why M has to be a subset of L, therefore $L = M$.

2. Let $K \subseteq F$ be a field extension and let $u \in F$. Assume that u is algebraic over K. Let L be the intersection of all subfields L of F such that $K \subseteq L$ and $u \in L$. Let

$$M = \{f(u) \in F : f(x) \in K[x]\}.$$

Prove that $L = M$. Use the hint for Problem 1.

3. Let $K = \mathbb{Q}$, $F = \mathbb{Q}[2^{1/3}, \xi]$ where $\xi = -(1/2) + (\sqrt{3}/2)i$ is a primitive 3^{rd} root of unity. Using the explicit form of Galois group which I distributed in class, prove that $\mathrm{Gal}_K(F) \cong D_3$.

4. Let K and F be as in Problem 3. Let $u = 2^{1/3} + 2^{2/3} + \xi$. Find $\mathrm{irr}_{\mathbb{Q}}(u)$. Flesh out the following outline of the solution:

Write $\mathrm{Gal}_K(F) = \{f_1, f_2, f_3, f_4, f_5, f_6\}$. Form the polynomial

$$p(x) = \prod_{i=1}^{6} (x - f_i(u)).$$

Then $p(x) \in \mathbb{Q}[x]$ and $p(u) = 0$, therefore $\mathrm{irr}_{\mathbb{Q}}(u) | p(x)$. Since u does not belong to any proper subfield of F, $p(x) = \mathrm{irr}_{\mathbb{Q}}(u)$. Explain why this last sentence constitutes a valid argument and prove carefully that u does not belong to any of the proper subfields of F, which are computed in the Galois Correspondence Class Handout.

5. Let $K = \mathbb{Q}$, $F = \mathbb{Q}[3^{1/4}, i]$.

(a) Show that F is a splitting field over K.

(b) Work out the Galois Correspondence between intermediate fields L such that $K \subseteq L \subseteq F$ and subgroups H of $\mathrm{Gal}_K(F)$ as I did in the class handout, identifying any normal subfield extensions and normal subgroups. Note that to prove that a subfield is a normal field extension it suffices to show that it is a splitting field of an appropriate polynomial in $K[x]$, and to prove that a subgroup is normal it simply must correspond under the priming operation to a normal subfield extension. Moreover, to prove that a subfield L is not a normal field extension one must identify an irreducible polynomial in $K[x]$ with at least one but not all roots in L.

6. Given a generic non-zero $a + b2^{1/3} + c2^{2/3} \in \mathbb{Q}[2^{1/3}]$ where $a, b, c \in \mathbb{Q}$, find $f, g, h \in \mathbb{Q}$ such that $(a + b2^{1/3} + c2^{2/3})^{-1} = f + g2^{1/3} + h2^{2/3}$. Each of the coefficients f, g, h should be expressed in terms of a, b, and c. Flesh out the following outline of the solution (choosing either Method I or Method II):

Method I: The field is isomorphic to the quotient ring $\mathbb{Q}[x]/I$ where $I = (x^3 - 2)$. The typical element in the quotient ring is $I + ax^2 + bx + c$ where $a, b, c \in \mathbb{Q}$. The division algorithm yields

$$x^3 - 2 = (ax^2 + bx + c)q(x) + r(x)$$

where the polynomials $q(x)$ and $r(x)$ have coefficients which are all expressions in a, b, c and $r(x)$ has degree ≤ 1. Hence

$$(I + ax^2 + bx + c)(I + q(x)) = I - r(x).$$

The division algorithm yields

$$x^3 - 2 = r(x)s(x) + t$$

where the polynomials $r(x)$, $s(x)$, and t have coefficients which are all expressions in a, b, c. Hence

$$(I - r(x))(I + s(x)) = I + t.$$

Putting everything together,

$$(I + ax^2 + bx + c)(I + q(x))(I + s(x)) = I + t,$$

therefore

$$(I + ax^2 + bx + c)^{-1} = (I + q(x))(I + s(x))(I + \frac{1}{t}) = I + hx^2 + gx + f$$

where the rational numbers f, g, h all have explicit formulas in terms of a, b, and c. Hence

$$(a + b2^{1/3} + c2^{2/3})^{-1} = f + g2^{1/3} + h2^{2/3}.$$

Method II: Let K and F be as in Problem 3, with $\mathrm{Gal}_K(F) = \{f_1, f_2, f_3, f_4, f_5, f_6\}$. Let

$$u = \prod_{i=1}^{6} f_i(a + b2^{1/3} + c2^{2/3}).$$

Then u is a rational number. Assuming that $f_1 = e$, we have

$$(a + b2^{1/3} + c2^{2/3})^{-1} = \frac{1}{u} \prod_{i=2}^{6} f_i(a + b2^{1/3} + c2^{2/3}).$$

Using complex conjugation as necessary, show that this product can be expressed as a linear combination 1, $2^{1/3}$, and $2^{2/3}$ using rational coefficients.