Math 223

Week 7 Solutions to Selected Problems

Section 12.3

23. The region of integration includes all $x \in [-3, 3]$. Given any one value of x in this range, y varies from 0 to $\sqrt{9 - x^2}$. So the region of integration is all points within and on the circle of radius 3 centered at the origin with $y \geq 0$. In terms of polar coordinates, θ ranges from 0 to π and r ranges from 0 to 3. So the polar coordinates evaluation of the double integral is

$$\int_0^{\pi} \int_0^3 r \sin(r^2)\ dr\ d\theta.$$

The antiderivative of $r \sin(r^2)$ is $-\frac{1}{2}\cos(r^2)$ using the u-substitution $u = r^2$, $du = 2r\ dr$.

25. The integral $\int_0^1 \int_y^{\sqrt{2-y^2}} (x + y)\ dx\ dy$ describes a Type II region which is bounded by the lines $y = 0$ and $y = 1$. For a given value of y between 0 and 1, x is ranging from a minimum value of y to a maximum value of $\sqrt{2 - y^2}$. Therefore the left hand boundary is given by the equation $x = y$ and the right hand boundary is given by the equation $x = \sqrt{2 - y^2}$. The region can be described as all points inside the circle of radius $\sqrt{2}$ and below the line $y = x$ and above the x-axis in the first quadrant. The angles in this region vary from 0 to $\frac{\pi}{4}$, and given any particular angle in this range, the radius is ranging from 0 to $\sqrt{2}$. So the integral in polar coordinates is

$$\int_0^{\frac{\pi}{4}} \int_0^{\sqrt{2}} r^2 \cos\theta + r^2 \sin\theta\ dr\ d\theta = \frac{2\sqrt{2}}{3}.$$

27. Let the surface of the pool be described by the equation $x^2 + y^2 = 20^2$ in the xy-plane where $z = 0$. Let south be in the direction of the positive x-axis and let east be in the direction of the positive y-axis (viewing the surface of the pool in the xyz coordinate system). Then the furthest point south on the surface of the pool is the point $(20, 0, 0)$ and the furthest point north on the surface of the pool is $(-20, 0, 0)$. The base of the pool can be described as a plane that passes through the three points $(-20, 0, -2)$, $(0, 20, -4.5)$, $(20, 0, -7)$. The equation of this plane is $ax + by + cz = d$. We may as well set $d = 1$, since we know the plane does not pass through the origin.

Plugging in these three points we quickly find that the equation of the plane is $z = -\frac{x+36}{8}$. So the volume of the pool is

$$\frac{1}{8}\int\int_R x + 36\ dA,$$

where R is the region $\{(x, y) : x^2+y^2 \leq 20^2\}$. In polar coordinates the region R can be expressed as $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 20$. so the volume of the pool is

$$V = \frac{1}{8}\int_0^{2\pi}\int_0^{20} r^2\cos\theta + 36r\ dr\ d\theta = 1800\pi \text{ cubic feet.}$$

Section 12.4

11. We are told that mass density is $\rho(x, y) = ky$ at position (x, y). The region of integration in polar coordinates can be described by $0 \leq \theta \leq \frac{\pi}{2}$, $0 \leq r \leq 1$. So the mass of this object is

$$\int_0^{\frac{\pi}{2}}\int_0^1 r \cdot kr\sin\theta\ dr\ d\theta = \frac{k}{3}.$$

The x-coordinate of the center of mass is the integral of x times the mass density over the mass, or

$$(\frac{k}{3})^{-1}\int_0^{\frac{\pi}{2}}\int_0^1 r \cdot r\cos\theta \cdot kr\sin\theta\ dr\ d\theta = (\frac{k}{3})^{-1}(\frac{k}{8}) = \frac{3}{8}.$$

The y-coordinate of the center of mass is the integral of y times the mass density over the mass, or

$$(\frac{k}{3})^{-1}\int_0^{\frac{\pi}{2}}\int_0^1 r \cdot r\sin\theta \cdot kr\sin\theta\ dr\ d\theta = (\frac{k}{3})^{-1}(\frac{k\pi}{16}) = \frac{3\pi}{16}.$$

18. We have

$$I_x = \int_0^2\int_0^2 y^2(1 + 0.1x)\ dy\ dx = 5.86667$$

and

$$I_y = \int_0^2\int_0^2 x^2(1 + 0.1x)\ dy\ dx = 6.13333,$$

so it's more difficult to rotate the blade about the y-axis.