Selected Solutions to Homework 5 Problems

Section 11.7

27. Given that $f(x,y) = x^4 + y^4 - 4xy + 2$, we have $\nabla f(x,y) = (4x^3 - 4y, 4y^3 - 4x)$. Solving $\nabla f(x,y) = (0,0)$ we obtain $x^3 = y$ and $y^3 = x$. Therefore $x = y^3 = (x^3)^3 = x^9$, $x(x^8 - 1) = 0$, $x = 0$ or $x = \pm 1$. Hence critical points are $(0,0)$, $(-1,-1)$, $(1,1)$. However, the only critical points we are interested in are those in the interior of the domain where $0 < x < 3$ and $0 < y < 2$. So we will just consider the stationary point $(1,1)$. If there's a max or a min value of f, it has to occur at $(1,1)$ or on the boundary of D. We have $f(1,1) = 0$.

Now consider $f(x,y)$ restricted to $x = 0$. We have $f(y) = y^4 + 2$. Since this is an increasing function of y, the minimum value must be at $y = 0$ and the maximum value must be at $y = 2$. We have $f(0) = 2$ and $f(2) = 18$.

Now consider $f(x,y)$ restricted to $x = 3$. We have $f(y) = y^4 - 12y + 83$. Critical point satisfies $0 = f'(y) = 4y^3 - 12$ or $y = 3^{1/3} = 1.44225$. Checking y values $0, 1.44225, 2$ we have $f(0) = 83$, $f(1.44225) = 70.0198$, $f(2) = 75$.

Now consider $f(x,y)$ restricted to $y = 0$. We have $f(x) = x^4 + 2$. Since this is an increasing function of x, the minimum value must be at $x = 0$ and the maximum value must be at $x = 3$. We have $f(0) = 2$ and $f(3) = 83$.

Now consider $f(x,y)$ restricted to $y = 2$. We have $f(x) = x^4 - 8x + 18$. Critical point satisfies $0 = f'(x) = 4x^3 - 8$ or $x = 2^{1/3} = 1.25992$. Checking x values $0, 1.25992, 3$ we have $f(0) = 18$, $f(1.25992) = 10.4405$, $f(3) = 75$.

Comparing all these outputs, the minimum is $f(1,1) = 0$ and the maximum is $f(3,0) = 83$.

29. Let $f(x,y) = -(x^2 - 1)^2 - (x^2y - x - 1)^2$. It's clear that $f(x,y) \leq 0$. It can only take on the value 0 when $x^2 - 1 = 0$ and $x^2y - x - 1 = 0$, otherwise we will get a negative value when we square each, add, then multiply by -1. This occurs when $x^2 = 1$. When $x = 1$ we have $y = 2$ and when $x = -1$ we have $y = 0$. So we have found the globally maximal value of f and two corresponding critical points.

To see that these are the only critical points, let $g(u,v) = -u^2 - v^2$. If we set $u = x^2 - 1$ and $v = x^2y - x - 1$, then by the chain rule $f_x = g_u u_x + g_v v_x$ and $f_y = g_u u_u + g_v v_y$. In matrix form this reads

$$\begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix} \begin{bmatrix} g_u \\ g_v \end{bmatrix} = \begin{bmatrix} 2x & 2xy - 1 \\ 0 & x^2 \end{bmatrix} \begin{bmatrix} -2u \\ -2v \end{bmatrix}.$$

The only way to obtain $\begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is to have

$$\begin{vmatrix} 2x & 2xy-1 \\ 0 & x^2 \end{vmatrix} = 0 \qquad \text{or} \qquad \begin{bmatrix} -2u \\ -2v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Now

$$\begin{bmatrix} -2u \\ -2v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

forces $u = v = 0$, which forces $x^2 - 1 = 0$ and $x^2y - x - 1 = 0$, and we have already treated this case. The only other alternative is

$$\begin{vmatrix} 2x & 2xy-1 \\ 0 & x^2 \end{vmatrix} = 0,$$

or $x = 0$, which makes $u = -1$ and $v = -1$. But this implies

$$\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which is impossible. So we just have the two critical points.

37. Done in class.

43. Let the dimensions of the box be length x, width y, height z. The total surface area of a rectangular solid with these dimensions is $2xy + 2xz + 2yz$. Since the top is missing, the surface area of the box is just $xy + 2xz + 2yz$. Now we must minimize $xy+2xz+2yz$ subject to $xyz = 32000$. The gradient equation reads

$$(y + 2z, x + 2z, 2x + 2y) = \lambda(yz, xz, xy),$$

which implies

$$\frac{y+2z}{yz} = \frac{x+2z}{xz} = \frac{2x+2y}{xy} = \lambda.$$

In other words,

$$\frac{1}{z} + \frac{2}{y} = \frac{1}{z} + \frac{2}{x} = \frac{2}{y} + \frac{2}{x}.$$

This in turn implies

$$x = y = 2z.$$

Combined with $xyz = 32000$ this implies $4z^3 = 32000$ or $z = 20$. Hence the minimal surface area results from a box with dimensions $(x, y, z) = (40, 40, 20)$. This surface area is $xy + 2xz + 2yz = 4800$.

Section 11.8

25. We want to minimize $f(x, y, z) = (x-2)^2 + (y-1)^2 + (z+1)^2$ on the level surface $g(x, y, z) = 1$, where $g(x, y, z) = x + y - z$. We wish to solve the two simultaneous equations $\nabla f = \lambda \nabla g$ and $x + y - z = 1$. This yields

$$2(x-2) = \lambda$$

$$2(y-1) = \lambda$$

$$2(z+1) = -\lambda$$

$$x + y - z = 1.$$

The first two equations imply that $x - 2 = y - 1$ or $y = x - 1$. The first and third equations imply that $x-2 = -z-1$ or $z = -x+1$. The fourth equation implies that $x+(x-1)-(-x+1) = 1$ or $x = 1$. Therefore $(x, y, z) = (1, 0, 0)$ and the minimum distance is $\sqrt{(1-2)^2 + (0-1)^2 + (0+1)^2} = \sqrt{3}$.

31. We want to maximize LWH subject to $9(L/2)^2 + 36(W/2)^2 + 4(H/2)^2 = 36$ (see hints on website). Setting $f(L, W, H) = LWH$ and $g(L, W, H) = 9(L/2)^2 + 36(W/2)^2 + 4(H/2)^2$ we want to see where $\nabla f = \lambda \nabla g$. This requires

$$WH = 9\lambda L$$

$$LH = 36\lambda W$$

$$LW = 4\lambda H.$$

Multiplying the first equation through by L, the second equation through by W, and the third equation through by H, we have

$$9\lambda L^2 = 36\lambda W^2 = 4\lambda H^2 = LWH.$$

Therefore $H^2 = \frac{9}{4}L^2$ and $W^2 = \frac{1}{4}L^2$. We know that we must have $9(L/2)^2 + 36(W/2)^2 + 4(H/2)^2 = 36$, so after the substitutions we obtain $27L^2 = 144$ or $L^2 = \frac{16}{3}$. This yields $W^2 = \frac{4}{3}$, $H^2 = 12$, $L^2W^2H^2 = \frac{256}{3}$, $LWH = \frac{16}{\sqrt{3}}$.