Selected Solutions to Homework 4 Problems

Section 11.5

23. Given that $\sqrt{xy} = 1 + x^2y$, we have $F(x,y) = 1$ where $F(x,y) = \sqrt{xy} - x^2y$. Therefore

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{\frac{1}{2}(xy)^{-\frac{1}{2}} \cdot y - 2xy}{\frac{1}{2}(xy)^{-\frac{1}{2}} \cdot x - x^2}.$$

29. We are given $r(t) = (\sqrt{1+t}, 2 + \frac{t}{3})$ and are asked to compute $f'(3)$, where $f(t) = T(r(t))$. By the chain rule, we have

$$f'(3) = (T_x(r(3), T_y(r(3)) \cdot r'(3) = (4,3) \cdot r'(3) = \cdots .$$

31. We can see that C is a function of T and D. We can also see that T and D are functions of time t, judging by the graphs provided. So conceptually we have

$$C = f(T,D) = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + 0.016D$$

and

$$C(t) = f(T(t), D(t)).$$

Therefore, by the chain rule,

$$C'(t) = f_T(T(t), D(t))T'(t) + f_D(T(t), D(t))D'(t).$$

The rate of change of speed at time $t = 20$ is therefore

$$C'(20) = f_T(T(20), D(20))T'(20) + f_D(T(20), D(20))D'(20).$$

We can see by the two graphs that $T(20) = 12.5$, $T'(20)$ is negative number (the slope of the tangent line, which can be approximated from the graph), $D(20) = 7$, and $D'(20)$ is a positive number. We can feed these values into the formula for $C'(20)$.

Section 11.6

26. $f(x, y) = 1000 - 0.005x^2 - 0.01y^2$, $\nabla f(x, y) = (-0.01x, -0.02y)$, $\nabla f(60, 40) = (-0.6, -0.8)$.

(a) $D_{(0,-1)} f(60, 40) = (-0.6, -0.8) \cdot (0, -1) = .8$. Ascend .8 meters per meter taken in southerly direction.

(b) $D_{(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})} f(60, 40) = (-0.6, -0.8) \cdot (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) = -.99$. Descend .99 meters per meter taken in northwesterly direction.

(c) The slope of the hill is greatest when we walk in the direction of greatest change in height, namely in the direction $(-0.6, -0.8)$. If we walk a small distance away from $(60, 40, 966)$ to $(60 + dx, 40 + dy, 966 + \Delta z)$, where $dx = -0.6h$ and $dy = -0.8h$, then we can approximate Δz via

$$\Delta z \approx dz = z_x(60, 40)dx + z_y(60, 40)dy = (-0.6)(-0.6h) + (-0.8)(-0.8h) = h.$$

The displacement from $(60, 40, 966)$ to $(60 - .6h, 50 - .8h, 966 + h)$ is $(-.6h, -.8h, h) = h(-.6, -.8, 1)$, so we are walking in 3 dimensions in direction $(-.6, -.8, 1)$. The angle that $(-.6, -.8, 1)$ makes with the horizontal is the angle between the vectors $(-.6, -.8, 1)$ and $(-.6, -.8, 0)$, which is $\cos^{-1}\left(\frac{(-.6,-.8,1)\cdot(-.6,-.8,0)}{|(-.6,-.8,1)||(-.6,-.8,0)|}\right) = \cos^{-1}\frac{1}{\sqrt{2}} = 45°$.

37. We know that the gradient vector is perpendicular to the tangent vector of a level curve. The level curve $x^2 + y^2 = 8$ has gradient vector $(2x, 2y)$, which at position $(x, y) = (2, 1)$ is equal to $\nabla f = (4, 2)$. If (x, y) is any point on the tangent line, then the displacement from $(2, 1)$ to (x, y) is $(x - 2, y - 1)$. This displacement must be perpendicular to $(4, 2)$. Hence $(4, 2) \cdot (x - 2, y - 1) = 0$. The tangent line equation is therefore $4(x - 2) + 2(y - 1) = 0$.

45. Using the outline in the hints, we have $f(x, y, z) = z - x^2 - y^2$ and $g(x, y, z) = 4x^2 + y^2 + z^2$. Therefore $\nabla f = (-2x, -2y, 1)$, $\nabla f(-1, 1, 2) = (2, -2, 1)$, $\nabla g = (8x, 2y, 2z)$, $\nabla g(-1, 1, 2) = (-8, 2, 4)$. Setting the tangent vector to (a, b, c) we want to solve $(2, -2, 1) \cdot (a, b, c) = 0$ and $(-8, 2, 4) \cdot (a, b, c) = 0$. Hence

$$2a - 2b + c = 0, \qquad -8a + 2b + 4c = 0.$$

Rearrange this to

$$2a - 2b = -c, \qquad -8a + 2b = -4c.$$

Treating c as a constant, and solving for a and b, we obtain $a = \frac{5}{6}c$ and $b = \frac{4}{3}c$. So the tangent vector is $(\frac{5}{6}c, \frac{4}{3}c, c) = c(\frac{5}{6}, \frac{4}{3}, 1)$. The direction vector of the tangent line can be taken to be any multiple of $(\frac{5}{6}, \frac{4}{3}, 1)$. Scaling up by a factor of 6 we choose $(5, 8, 6)$. So one possible equation of the tangent line is

$$r(t) = (-1, 1, 2) + t(5, 8, 6) = (5t - 1, 8t + 1, 6t + 2).$$