Selected Solutions to Homework 2 Problems

Section 10.9

8. Given $r(t) = (2\cos t, 3t, 2\sin t)$, velocity is $r'(t) = (-2\sin t, 3, 2\cos t)$, speed is $||r'(t)|| = \sqrt{13}$, and acceleration is $r''(t) = (-2\cos t, 0, -2\sin t)$.

15. $r(t) = (t^2, 5t, t^2 - 16t)$, $r'(t) = (2t, 5, 2t - 16)$, speed $= \frac{ds}{dt} = |r'(t)| = \sqrt{4t^2 + 25 + 4t^2 - 64t + 256} = \sqrt{8t^2 - 64t + 281}$, rate of change of speed is $\frac{d^2s}{ds^2} = (\sqrt{8t^2 - 64t + 281})' = \frac{1}{2}(8t^2 - 64t + 281)^{-1/2}(16t - 64)$, when speed is minimized its rate of change is zero, which occurs when $16t - 64 = 0$ or $t = 4$.

22. Let $r(t) = (x(t), y(t))$ be the position of the shell at time t second after firing. Starting with $r''(t) = (0, -9.81)$, which we take as an axiom (acceleration due to gravity is 9.81 meters per second per second towards earth), we have $r'(t) = (0, -9.81)t + C$ where C is a vector constant. Given that muzzle speed is v_0 meters per second at time zero, then $r'(0) = v_0(\cos 30°, \sin 30°) = v_0(\frac{\sqrt{3}}{2}, \frac{1}{2})$. This yields $C = v_0(\frac{\sqrt{3}}{2}, \frac{1}{2})$. So now we have $r'(t) = (0, -9.81)t + v_0(\frac{\sqrt{3}}{2}, \frac{1}{2})$. Therefore $r(t) = (0, -4.905)t^2 + v_0 t(\frac{\sqrt{3}}{2}, \frac{1}{2}) + D$ where D is a vector constant. Assuming $r(0) = (0, 0)$, this implies $D = (0, 0)$. Hence $r(t) = (0, -4.905)t^2 + v_0 t(\frac{\sqrt{3}}{2}, \frac{1}{2})$. We are told that maximum height of the shell is 500 meters. Translated into mathematical terms, this says that if T is the time at which maximum height is achieved, then $y(T) = 500$ when $y'(T) = 0$. This yields two equations:

$$-4.905T^2 + \frac{v_0 T}{2} = 500,$$

$$-9.81T + \frac{v_0}{2} = 0.$$

The second equation tells us that we can replace $\frac{v_0}{2}$ in the first equation by $9.81T$. Making this substitution, the first equation says

$$-4.905T^2 + 9.81T^2 = 500,$$

$$4.905T^2 = 500,$$

$$T = \sqrt{\frac{500}{4.905}} = 10.0964.$$

So maximum height occurs a little later than 10 seconds into the flight. This implies

$$\frac{v_0}{2} = 9.81T = 99.0454,$$

therefore $v_0 = 198.091$ meters per second.

23. Done in class.

24. $r''(t) = (0, -32)$, hence $r'(t) = (0, -32)t + (a, b)$. The initial conditions say $r'(0)$ has length 115 and angle 50 degrees, therefore $(a, b) = r'(0) = (115\cos 50°, 115\sin 50°) = (73.9, 88.1)$. Hence $r'(t) = (0, -32)t + (73.9, 88.1)$. Now we obtain $r(t) = (0, -32)\frac{t^2}{2} + (73.9, 88.1)t + (p, q)$. The initial conditions say $r(0) = (0, 3)$, which implies $(p, q) = (0, 3)$ and $r(t) = r(t) = (0, -32)\frac{t^2}{2} + (73.9, 88.1)t + (0, 3) = (73.9t, -16t^2 + 88.1t + 3)$. The ball travels 400 feet horizontally when $73.9t = 400$ or $t = 5.4$ seconds. In this time the height of the ball is $-16(5.4)^2 + 88.1(5.4) + 3 = 12.2$ feet above ground. So yes, the ball clears the fence.

Section 11.1

21. The point $(-3, 3)$ is somewhere between the contour line $z = 60$ and the contour line $z = 50$, so I will guess that $f(-3, 3) = 55$. The point $(3, -2)$ is on the contour line $z = 40$, so $f(3, -2) = 40$. When you lift these contour lines to their appropriate height, it seems as if the graph is closer to the z axis as z increases. So I'm visualizing a surface that looks like a tube of toothpaste that has been squeezed narrower and narrower the higher up you go, or maybe a burning candle which is sitting in a puddle of melted wax which is accumulating at the bottom.

35. The isothermals are at all points (x, y, z) where $z = T(x, y) = k$. Therefore $100/(1 + x^2 + 2y^2) = k$, or $x^2 + 2y^2 = (100/k) - 1$. Hence the level curves are ellipses. As k increases, the ellipses decrease in size. The largest value of k is 100, which occurs at the point $(0, 0, 100)$.

Section 11.2

13. Using polar coordinates, $x = r\cos\theta$ and $y = r\sin\theta$. As $(x, y) \to (0, 0)$ we must have $r \to 0$. We have

$$\frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1} = \frac{r^2}{\sqrt{r^2 + 1} - 1} = \frac{s}{\sqrt{s + 1} - 1}$$

where $s = r^2$. As $r \to 0$, the last expression is indeterminate of the form $\frac{0}{0}$. Using L'Hopital's rule,

$$\lim_{s\to 0} \frac{s}{\sqrt{s+1}-1} = \lim_{s\to 0} \frac{1}{\frac{1}{2\sqrt{s+1}}} = 2.$$

So my conclusion is that

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} = 2.$$

23. As we learn in Calc I, the natural log function is continuous at all points in its domain, which is the set of positive numbers. So we must have $x^2+y^2 > 4$, or all points strictly more than 2 units from the origin.

29. Write $x = r\cos\theta$ and $y = r\sin\theta$. Then

$$\frac{x^3+y^3}{x^2+y^2} = \frac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2} = r(\cos^3\theta + \sin^3\theta).$$

As (x, y) approaches the origin, the angle θ may vary, but $r \to 0^+$. The value of $\cos^3\theta + \sin^3\theta$ is always between -2 and 2. So we have

$$-2r \leq \frac{x^3+y^3}{x^2+y^2} \leq 2r.$$

Now let $r \to 0^+$. Then $-2r \to 0$ and $2r \to 0$, hence by the Squeeze Principle (Calc I concept) $\frac{x^3+y^3}{x^2+y^2} \to 0$.