Math 223

Week 12 Solutions to Selected Problems

**Section 13.4**

23. The area of the triangle is $A = \frac{1}{2}$. The triangle has 3 sides, $C_1, C_2, C_3$, which can be parameterized in the positive direction via

$$r_1(t) = (t, 0), \qquad r_2(t) = (1-t, t), \qquad r_3(t) = (0, 1-t), \qquad 0 \le t \le 1.$$

Therefore

$$\overline{x} = \frac{1}{2A} \oint_C x^2\, dy = \frac{1}{2A} \oint_{C_1} x^2\, dy + \frac{1}{2A} \oint_{C_2} x^2\, dy + \frac{1}{2A} \oint_{C_3} x^2\, dy =$$

$$\int_0^1 x_1(t)^2 y_1'(t)\, dt + \int_0^1 x_2(t)^2 y_2'(t)\, dt + \int_0^1 x_3(t)^2 y_3'(t)\, dt =$$

$$\int_0^1 t^2 \cdot 0\, dt + \int_0^1 (1-t)^2 \cdot 1\, dt + \int_0^1 0^2 \cdot (-1)\, dt = \frac{1}{3}.$$

**Section 13.5**

17. By Theorem 11, the divergence of the curl of a vector field is equal to 0 everywhere (reason: mixed partial derivatives are equal, so everything cancels out in this computation). So if curl $G = (xy^2, yz^2, zx^2)$ then we would expect the divergence of $(xy^2, yz^2, zx^2)$ to be equal to zero everywhere. But the divergence is $y^2 + z^2 + x^2$, which is only equal to 0 at the origin. So no, curl $G = (xy^2, yz^2, zx^2)$ is not possible.

25. Let $F = (P, Q, R)$ and $G = (p, q, r)$. Then

$$F \times G = (Qr - qR, -Pr + pR, Pq - pQ).$$

The divergence of this is

$$Q_x r + Q r_x - q_x R - q R_x - P_y r - P r_y + p_y R + p R_y + P_z q + P q_z - p_z Q - p Q_z.$$

We can organize this into

$$p(R_y - Q_z) + q(-R_x + P_z) + r(Q_x - P_y) - P(r_y - q_z) - Q(-r_x + p_z) - R(q_x - p_y)$$

which is equal to $G \cdot \text{curl } F - F \cdot \text{curl } G$.

29.(a) We have $\mathbf{r} = (x, y, z)$ and $r = (x^2 + y^2 + z^2)^{1/2}$. Therefore

$$\nabla r = \left( \frac{x}{(x^2+y^2+z^2)^{1/2}}, \frac{y}{(x^2+y^2+z^2)^{1/2}}, \frac{z}{(x^2+y^2+z^2)^{1/2}} \right) = \frac{\mathbf{r}}{r}.$$