Math 223 Final Exam Solutions Fall 2010 Name:

Show all work

Note: Problems 1–4 refer to the surface S described below and depicted in the margin. Problem 4 refers to the curve C which bounds S and is depicted in the margin. Problem 5 refers to the surface T described below and depicted in the margin.

Surfaces: S is the portion of the cone $x = \sqrt{y^2 + z^2}$ that lies between the planes $x = 0$ and $x = 5$ and above the xy-plane. T is the closed surface that results by sealing off the surface S by the planes $z = 0$ and $x = 5$.

1. Find a parameterization of S in terms of the two parameters r and θ, including the inequalities these parameters satisfy.

Solution: $x = r$, $y = r\cos\theta$, $z = r\sin\theta$, $0 \le r \le 5$, $0 \le \theta \le \pi$.

2. Compute the mass of the surface S, given that it has mass density $\rho(x, y, z) = x$ grams per square centimeter at position (x, y, z). Assume that x, y, and z have units of centimeters.

Solution:

$$M = \int\int_S \rho \, dS = \int_0^5 \int_0^\pi \rho(r, r\cos\theta, r\sin\theta)||r_r \times r_\theta|| \, d\theta \, dr$$

$$\rho(r, r\cos\theta, r\sin\theta) = r$$

$$||r_r \times r_\theta|| = \left|\left|\det \begin{pmatrix} i & j & k \\ 1 & \cos\theta & \sin\theta \\ 0 & -r\sin\theta & r\cos\theta \end{pmatrix}\right|\right| = ||(r, -r\cos\theta, -r\sin\theta)|| = \sqrt{2}r$$

$$M = \int_0^5 \int_0^\pi \sqrt{2}r^2 \, d\theta \, dr = \sqrt{2}\pi\frac{5^3}{3} \approx 185.12 \text{ grams.}$$

3. Compute the flux of the vector field $F(x, y, z) = (y, z, x)$ across the surface S using normal vectors which point in the negative x direction. Assume that F measures the vector velocity of a current in feet per second and that x, y, and z have units of feet.

Solution:

$$\text{Flux} = \int\int_S F \cdot dS = -\int_0^5 \int_0^\pi F(r, r\cos\theta, r\sin\theta) \cdot (r_r \times r_\theta)\; d\theta\; dr$$

$$F(r, r\cos\theta, r\sin\theta) \cdot (r_r \times r_\theta) = (r\cos\theta, r\sin\theta, r) \cdot (r, -r\cos\theta, -r\sin\theta) =$$
$$r^2\cos\theta - r^2\sin\theta\cos\theta - r^2\sin\theta = r^2(\cos\theta - \sin\theta\cos\theta - \sin\theta)$$

Inner Integral:

$$\int_0^\pi r^2(\cos\theta - \sin\theta\cos\theta - \sin\theta)\; d\theta = r^2(\sin\theta - \frac{1}{2}\sin^2\theta + \cos\theta)\Big|_0^\pi = -2r^2$$

Outer Integral: $2\int_0^5 r^2\; dr = \frac{250}{3}$ feet^3 per second.

4. Stoke's Theorem equates work done by a vector field F around a closed curve C to the flux of curl F across a surface that C bounds. Our surface S is bounded by C, where C consists of two straight lines in the xy-plane and a semicircle. Using Stoke's Theorem, calculate the work done by the vector field $(z, 0, y^2)$ around C in the counter-clockwise direction relative to the positive x-axis by computing the appropriate surface integral.

Solution:

$$W = \int_C F \cdot dr = \int \int_S \text{curl } F \cdot dS =$$

$$\int_0^5 \int_0^\pi \text{curl } F(r, r\cos\theta, r\sin\theta) \cdot (r_r \times r_\theta)\ d\theta\ dr$$

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial_x} & \frac{\partial}{\partial_y} & \frac{\partial}{\partial_z} \\ z & 0 & y^2 \end{vmatrix} = (2y, 1, 0)$$

$$\text{curl } F(r, r\cos\theta, r\sin\theta) \cdot (r_r \times r_\theta) = (2r\cos\theta, 1, 0) \cdot (r, -r\cos\theta, -r\sin\theta) =$$

$$(2r^2 - r)\cos\theta$$

$$W = \int_0^5 \int_0^\pi (2r^2 - r)\cos\theta\ d\theta\ dr = 0.$$

5. The Divergence Theorem equates flux across a closed surface to a triple integral over the region it bounds. Using the Divergence Theorem, (a) explain why the flux of the vector field $F = (x, y, 0)$ across the surface T is equal to twice the volume bounded by T, and (b) compute the flux using an appropriate integral of your choice.

Solution:

(a) The divergence of $(x, y, 0)$ is 2. The triple integral of 2 over the region bounded by T is 2 times the volume of T.

(b) The region bounded by the full cone of which T is half can be generated by revolving the line $z = x$ about the x-axis, $0 \leq x \leq 5$. So its volume is $\int_0^5 \pi x^2 \, dx = \frac{\pi 5^3}{3}$.