Math 223 Exam 3 Solutions Fall 2010 Name:

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1. Let S represent the sphere of radius 1 centered about the origin, with equation $x^2 + y^2 + z^2 = 1$. Let C represent the inverted cone with vertex at the origin, with equation $z = \sqrt{3x^2 + 3y^2}$. Let E represent the region inside S and above C. Assuming that the mass density of this region is $\rho(x, y, z) = z$ grams per cubic centimeter at position (x, y, z) and that x, y, z are measured in units of centimeters, find the mass of E. Use spherical coordinates.

Solution: Using spherical coordinates we can see that $0 \le \phi \le 1$, $0 \le \theta \le 2\pi$, and that the integrand is

$$z\rho^2 \sin\phi = \rho^3 \cos\phi \sin\phi.$$

We just need to see how ϕ is varying. In the yz plane the surfaces satisfy $y^2 + z^2 = 1$ and $z^2 = 3y^2$. This implies $y = \frac{1}{2}$ and $z = \frac{\sqrt{3}}{2}$. Hence the radial line makes an angle of $\frac{\pi}{3}$ with the y-axis, $\frac{\pi}{6}$ with this z axis. Hence $0 \le \phi \le \frac{\pi}{6}$. So the mass is

$$M = \int_0^1 \int_0^{2\pi} \int_0^{\frac{\pi}{6}} \rho^3 \cos\phi \sin\phi \; d\phi \; d\theta \; d\rho.$$

Inner integral: $\frac{1}{2}\rho^3 \sin^2\theta\Big|_0^{\frac{\pi}{6}} = \frac{1}{8}\rho^3$.

Middle integral: $\frac{1}{8}\rho^3\theta\Big|_0^{2\pi} = \frac{1}{4}\rho^3\pi$.

Outer integral: $\frac{1}{16}\rho^4\pi\Big|_0^1 = \frac{\pi}{16}$.

2. Compute the line-integral of the scalar field $f(x, y) = x^2$ over the portion of the unit circle which runs counterclockwise from $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ to $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$.

Solution: Using $x(t) = \cos t$, $y(t) = \sin t$, $\frac{\pi}{3} \leq t \leq \frac{4\pi}{3}$, we have

$$\int_C f \; ds = \int_{\frac{\pi}{3}}^{\frac{4\pi}{3}} \cos^2 t \sqrt{(-\sin t)^2 + (\cos t)^2} \; dt = \frac{1}{2} \int_{\frac{\pi}{3}}^{\frac{4\pi}{3}} 1 + \cos 2t \; dt =$$

$$\frac{1}{2}(t + \frac{1}{2} \sin 2t)\Big|_{\frac{\pi}{3}}^{\frac{4\pi}{3}} = \frac{\pi}{2}.$$

3. Compute the work done by the vector field $f(x, y) = (y, x)$ along the straight line path starting at $(1, 2)$ and ending at $(2, 1)$.

Solution: Using $x(t) = 1 + t$, $y(t) = 2 - t$, $0 \leq t \leq 1$, we have

$$\int_C f \cdot dr = \int_0^1 (2 - t, 1 + t) \cdot (1, -1)\ dt = \int_0^1 1 - 2t\ dt = t - t^2\Big|_0^1 = 0.$$

Alternate Solution: Note that the vector field is the gradient of xy, hence is conservative. Therefore

$$W = xy\Big|_{(1,2)}^{(2,1)} = (2)(1) - (1)(2) = 0.$$

4. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defined by

$$F(x, y) = y\cos(xy)\ \overrightarrow{i} + (x\cos(xy) + e^y)\ \overrightarrow{j}.$$

(a) Prove that $F(x, y)$ is conservative.

(b) Compute the work done by $F(x, y)$ along the curve $x(t) = \cos t$, $y(t) = \sin t$, $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$.

Solution: Using $P = y\cos(xy)$, $Q = x\cos(xy) + e^y$, we have

$$P_y = Q_x = \cos(xy) - yx\sin(xy).$$

Hence F is conservative. Solving $F = \nabla f$, we have $y\cos(xy) = f_x$ and $x\cos(xy) + e^y = f_y$, therefore $f = \sin(xy) + e^y$. The endpoints of the path are $(0, -1)$ and $(0, 1)$. Therefore the work integral is

$$f(0, 1) - f(0, -1) = e - \frac{1}{e}.$$

5. Using Green's Theorem, compute the area bounded by the radial line $\theta = a$, the radial line $\theta = b$, and the circle $x^2 + y^2 = 1$.

Solution: We will compute the work done by the vector field $(-\frac{y}{2}, \frac{x}{2})$ around the boundary of the region in the counter-clockwise direction.

Along the first radial line: $x(t) = t\cos a$, $y(t) = t\sin a$, $0 \le t \le 1$.

$$W_1 = \int_0^1 (\frac{-t\sin a}{2}, \frac{t\cos a}{2}) \cdot (\cos a, \sin a)\ dt = 0.$$

Along the circle between the radial lines: $x(t) = \cos t$, $y(t) = \sin t$, $a \le t \le b$.

$$W_2 = \int_a^b (\frac{-\sin t}{2}, \frac{\cos t}{2}) \cdot (-\sin t, \cos t)\ dt = \int_a^b \frac{1}{2}\ dt = \frac{b-a}{2}.$$

Along the second radial line: $x(t) = t\cos b$, $y(t) = t\sin b$, $0 \le t \le 1$.

$$W_3 = -\int_0^1 (\frac{-t\sin b}{2}, \frac{t\cos b}{2}) \cdot (\cos b, \sin b)\ dt = 0.$$

Hence the area is $W_1 + W_2 + W_3 = \frac{b-a}{2}$.