Math 223 Exam 2 Fall 2010 Name:

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1. (20 points) Let $f(x, y) = xy(x^2 - 1)$.

(a) Find all critical points and classify them using the Second Derivatives test (if possible).

(b) Find the maximum and minimum output of $f(x, y)$ restricted to all inputs (x, y) within the triangle with vertices $(0, 0)$, $(1, 1)$, and $(0, 1)$.

Solutions: (a) We have $f(x, y) = x^3y - xy$, therefore $f_x = 3x^2y - y$ and $f_y = x^3 - x$. The critical points occur where

$$\begin{aligned} y(3x^2 - 1) &= 0, \\ x(x^2 - 1) &= 0. \end{aligned}$$

The second equation is satisfied if and only if $x = 0$ or $x = 1$ or $x = -1$. Given this, the first equation is satisfied if and only if $y = 0$. So the three critical points are $(0, 0)$, $(1, 0)$, and $(-1, 0)$. The discriminant of f is

$$f_{xx}f_{yy} - f_{xy}^2 = (6xy)(0) - (3x^2 - 1)^2 = -(3x^2 - 1)^2 < 0$$

at the three inputs. Hence all three critical points are saddle points.

(b) The first two critical points of f lie on the boundary of the triangle, and the third one is irrelevant. So the only places we need check are on the boundary. The boundary is contained in the union of the lines lines $x = 0$, $y = 1$, and $x = y$. Consider the cases:

$x = 0$: The function outputs 0 along this boundary.

$y = 1$: The function outputs $x^3 - x$ at the point $(x, 1)$. Critical point is $(t, 1)$ where $3t^2 - 1 = 0$ or $t = \sqrt{\frac{1}{3}}$. We have $f(\sqrt{\frac{1}{3}}, 1) = \frac{-2}{3}\sqrt{\frac{1}{3}} \approx -.3849$ at this point and $f(0, 1) = f(1, 1) = 0$.

$y = x$: The function outputs $x^4 - x^2$ at the point (x, x). Critical point is (t, t) where $4t^3 - 2t = 0$ or $t = 0, \sqrt{\frac{1}{2}}$. We have $f(0, 0) = f(1, 1) = 0$ and $f(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}) = -\frac{1}{4} = -.25$.

Conclusion: maximum output is 0, minimum output is $-\frac{2}{3\sqrt{3}}$.

2. (20 points) Using the method of Lagrange Multipliers, find the maximum value of xyz given that $x - y + z = 1$.

Solution: This is a bit of a trick question in that no maximum value is attained (the value of xyz is t^2 at the point $(t, t, 1)$, and t can be arbitrarily large). However, going through the motions, we have $f(x, y, z) = xyz$, $g(x, y, z) = x - y + z$,

$$\nabla f = \lambda \nabla g \Rightarrow (yz, xz, xy) = \lambda(1, -1, 1) \Rightarrow$$

$$(x, y, z) \in \{(1, 0, 0), (0, -1, 0), (0, 0, 1), (\frac{1}{3}, -\frac{1}{3}, \frac{1}{3})\}.$$

The maximum value of xyz given these inputs is 0 and the minimum value of xyz given these inputs is $-\frac{1}{27}$.

All we have really proved here is that ∇f is parallel to ∇g at these inputs. But setting $y = x + z - 1$ and $F(x, z) = xyz = xz(x + z - 1)$ one can readily check that $(x, z) = (1, 0), (0, 0), (0, 1), (\frac{1}{3}, \frac{1}{3})$ are critical points of F, not necessarily locations of extrema.

3. (20 points) Exchange the order of integration in the double integral

$$\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \int_{-\sqrt{1-x^2}}^{-\frac{1}{2}} f(x,y)\ dy\ dx.$$

Solution: The region of integration is currently described as a Type I region, the portion of a unit circle below the line $y = -\frac{1}{2}$. The Type II description of this region has y varying from -1 to $-\frac{1}{2}$ and x varying from the left boundary $x = -\sqrt{1-y^2}$ to the right boundary $x = \sqrt{1-y^2}$. The Type II evaluation yields

$$\int_{-1}^{-\frac{1}{2}} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\ dx\ dy.$$

4. (20 points) Let R be a metal plate covering the region of the $\mathbb{R}^2$ consisting of all the points on and below the line $y = x$ which fall inside the circle of radius 1 centered at $(1, 0)$. Find the mass of R, given that its mass density function is $\rho(x, y) = y$. Use polar coordinates to evaluate the double integral.

Solution: The angles in this region vary from $\theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{4}$. The equation of the circle is $(x-1)^2 + y^2 = 1$ or $x^2 + y^2 = 2x$ or $r^2 = 2r\cos\theta$ or $r = 2\cos\theta$. The radius in the region varies from 0 to $2\cos\theta$ along the radial line of angle θ. Therefore the polar coordinates evaluation of the mass is

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \int_0^{2\cos\theta} r \cdot r \sin\theta \; dr \; d\theta.$$

Inner integral: $\frac{2r^3}{3}\sin\theta\Big|_0^{2\cos\theta} = \frac{16}{3}\cos^3\theta\sin\theta$

Outer integral: making the substitution $u = \cos\theta$, $du = -\sin\theta\; d\theta$, the integral is

$$-\int_0^{\frac{\sqrt{2}}{2}} \frac{16}{3}u^3 \; du = -\frac{4}{3}u^4\Big|_0^{\frac{\sqrt{2}}{2}} = -\frac{16}{3}.$$

Since the mass density function takes on negative values, it's not too surprising that the total mass is negative.

5. Let E represent all the points in $\mathbb{R}^3$ within a sphere of radius 3 centered at the origin, i.e. all (x, y, z) such that $x^2 + y^2 + z^2 \leq 9$. Let R denote the region of the xy plane in the 1st quadrant between the radial lines $\theta = \frac{\pi}{6}$ and $\theta = \frac{\pi}{3}$, and let F denote the solid region in $\mathbb{R}^3$ consisting of all (x, y, z) such that $(x, y, 0)$ falls in R. Find the volume of the intersection of E and F by means of a triple integral calculation. You will find cylindrical coordinates useful here.

Solution: The region of intersection can be approximated by a bunch of pencils of varying length glued together, passing through the xy plane between the radial lines $\theta = \frac{\pi}{6}$ and $\theta = \frac{\pi}{3}$ anywhere from 0 to 3 units away from the origin and piercing the sphere below the xy plane and above the xy plane. So in terms of cylindrical coordinates, we have $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}$, $0 \leq r \leq 3$, and $-\sqrt{9 - r^2} \leq z \leq \sqrt{9 - r^2}$. Using symmetry, doubling the volume above the xy plane, the triple integral evaluation is

$$V = 2 \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \int_0^3 \int_0^{\sqrt{9-r^2}} r \cdot 1 \; dz \; dr \; d\theta.$$

Inner integral: $rz|_0^{\sqrt{9-r^2}} = r\sqrt{9 - r^2}$.

Middle integral: $-\frac{1}{3}(9 - r^2)^{\frac{3}{2}}\Big|_0^3 = 9$.

Outer Integral: $9\theta|_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{3\pi}{2}$.

Hence $V = 3\pi$.

Check: the volume of the sphere is $\frac{4}{3}\pi 3^3 = 36\pi$. F carves out $\frac{\frac{\pi}{3} - \frac{\pi}{6}}{2\pi} = \frac{1}{12}$ of the sphere. $\frac{36\pi}{12} = 3\pi$.

Spherical coordinates evaluation (one person attempted this):

$$V = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \int_0^{\pi} \int_0^3 \rho^2 \sin\phi \cdot 1 \; d\rho \; d\phi \; d\theta.$$