Math 223 Exam 1 Fall 2010 Name:

Show all work

1. (20 points) Let $r(t) = (t\cos t, t^2, t\sin t)$.

(a) Find the unit tangent vector at time $t = \frac{\pi}{2}$.

(b) Find the unit normal vector at time $t = \frac{\pi}{2}$.

(c) Find the curvature at time $t = \frac{\pi}{2}$.

Solution:

$$r(t) = (t\cos t, t^2, t\sin t)$$

$$r'(t) = (\cos t - t\sin t, 2t, \sin t + t\cos t)$$

$$r''(t) = (-2\sin t - t\cos t, 2, 2\cos t - t\sin t)$$

$$||r'(t)|| = \sqrt{(\cos t - t\sin t)^2 + (2t)^2 + (\sin t + t\cos t)^2} = \sqrt{1+5t^2}$$

$$T(t) = (1+5t^2)^{-\frac{1}{2}} r'(t)$$

$$T'(t) = -5t(1+5t^2)^{-\frac{3}{2}} r'(t) + (1+5t^2)^{-\frac{1}{2}} r''(t)$$

$$T(\frac{\pi}{2}) = (1+5(\frac{\pi}{2})^2)^{-\frac{1}{2}}(-\frac{\pi}{2}, \pi, 1) \approx (-0.43, 0.86, 0.27)$$

$$T'(\frac{\pi}{2}) = -5\frac{\pi}{2}(1+5(\frac{\pi}{2})^2)^{-\frac{3}{2}}(-\frac{\pi}{2}, \pi, 1) + (1+5(\frac{\pi}{2})^2)^{-\frac{1}{2}}(-2, 2, -\frac{\pi}{2}) \approx (-0.29, 0.041, -0.59)$$

$$N(\frac{\pi}{2}) = \frac{T'(\frac{\pi}{2})}{||T'(\frac{\pi}{2})||} \approx \frac{(-0.29, 0.041, -0.59)}{0.66} = (-0.44, -.06, -0.89)$$

$$\kappa(\frac{\pi}{2}) = \frac{||T'(\frac{\pi}{2})||}{||r'(\frac{\pi}{2})||} \approx .66/3.65 \approx 0.18$$

2. (20 points) Let $f(x,y) = \sqrt{x^2+y^2-1}$.

(a) Find the domain of this function.

(b) Sketch level curves corresponding to $z = 0$, $z = 3$, and $z = 5$ in the xy coordinate system.

(c) Graph $z = f(x,y)$ in the xyz coordinate system.

Solution: The domain requires $x^2+y^2 \geq 1$. The level curves are $x^2+y^2=1$, $x^2+y^2=10$, and $x^2+y^2=26$, or circles of radius 1, $\sqrt{10}$, and $\sqrt{26}$ respectively, centered at the origin. As z increases, the circles increase in radius, with the smallest circle of radius 1 in the xy plane, so the graph is an inverted cone with the bottom sliced out of it.

3. (20 points) Let

$$f(x,y) = \begin{cases} \frac{x}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Find all (x,y) where $f(x,y)$ is continuous. Explain carefully.

Solution: As a quotient of polynomial functions, $f(x,y)$ is continuous where $(x,y) \neq (0,0)$. The limit $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist, because the outputs are 0 along the y-axis away from the origin, and the outputs approach ∞ as inputs approach $(0,0)$ along the positive x-axis. Therefore the function is not continuous at $(0,0)$.

4. (20 points) Find the equation of the plane tangent to the surface $f(x,y) = (x-y+xy)^5$ at the point $(2,1,243)$.

Solution: The tangent plane equation is the same as the linear approximation, namely

$$z - 243 = f_x(2,1)(x-2) + f_y(2,1)(y-1).$$

We have

$$f_x(x,y) = 5(x-y+xy)^4(1+y)$$

and

$$f_y(x,y) = 5(x-y+xy)^4(-1+x),$$

therefore

$$f_x(2,1) = 810$$

and

$$f_y(2,1) = 405,$$

therefore the tangent plane equation is

$$z - 243 = 810(x-2) + 405(y-1).$$

5. (20 points) Let $f(x,y) = (x - y + xy)^5$. Using differentials, estimate the value of $f(.98, 1.03)$. Do not use a calculator for any of the computations.

Solution: Let $(1,1)$ be the reference input. That makes $dx = -.02$ and $dy = .03$. The linear approximation is therefore

$$f(.98, 1.03) \approx f(1,1) + f_x(1,1)(-.02) + f_y(1,1)(.03).$$

We have

$$\begin{aligned} f(1,1) &= 1, \\ f_x(x,y) &= 5(x - y + xy)^4(1 + y), \\ f_y(x,y) &= 5(x - y + xy)^4(-1 + x), \\ f_x(1,1) &= 10, \\ f_y(1,1) &= 0, \end{aligned}$$

therefore

$$f(.98, 1.03) \approx 1 + (10)(-.02) + (0)(.03) = 0.8.$$

Check: $(.98 - 1.03 + (.98)(1.03))^5 = 0.812828$.