A graph-theoretic method for choosing a spanning set for a finite-dimensional vector space

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Abstract

We describe a graph-theoretic method for finding a spanning set of vectors for a finite-dimensional vector space from among a set of vectors generated combinatorially, based on nonzero and sign pattern matrices which guarantee non-singularity. We use our method to identify spanning sets for a variety of subspaces of the Grossman-Larson-Wright rooted tree module. The existence of these spanning sets yields combinatorial proofs of special cases of the Jacobian conjecture.
Jacobian Conjecture


For each

\[(f_1, \ldots, f_n) \in \mathbb{C}[x_1, \ldots, x_n]^n\]

with

\[\det \left( \frac{\partial f_i}{\partial x_j} \right) \in \mathbb{C}^*\]

there exists

\[(g_1, \ldots, g_n) \in \mathbb{C}[x_1, \ldots, x_n]^n\]

such that

\[(g_1(f_1, \ldots, f_n), \ldots, g_n(f_1, \ldots, f_n)) = (x_1, \ldots, x_n).\]

Example:

\[(f_1, f_2) = (x_1 - (x_1 + ix_2)^2, x_2 - i(x_1 + ix_2)^2)\]

\[(g_1, g_2) = (x_1 + (x_1 + ix_2)^2, x_2 + i(x_1 + ix_2)^2)\]
Homogeneous Symmetric Reduction

Michiel de Bondt and Arno van den Essen, *A reduction of the Jacobian conjecture to the symmetric case*, Proc. Amer. Math. Soc. 133 (2005), no. 8, 2201-2205:

**Theorem:** *The Jacobian conjecture is true if it holds for all polynomial systems \((f_1, \ldots, f_n)\) having the form*

\[
(f_1, \ldots, f_n) = (x_1 - \frac{\partial P}{\partial x_1}, \ldots, x_n - \frac{\partial P}{\partial x_n}),
\]

*where \(P \in \mathbb{C}[x_1, \ldots, x_n]\) is a homogeneous polynomial. In fact, it suffices to prove the case \(\deg(P) = 4\).*

In our example, \(P = \frac{1}{3}(x_1 + ix_2)^3\).

**Note:** *Under the hypotheses of the homogeneous symmetric reduction of the Jacobian conjecture,*

\[
\left(\frac{\partial^2 P}{\partial x_i \partial x_j}\right)^n = 0.
\]
Inverse Expansion


**Theorem:** Let $P \in \mathbb{C}[x_1, \ldots, x_n]$ be homogeneous of degree $\geq 3$. The inverse of

$$(x_1 - \frac{\partial P}{\partial x_1}, \ldots, x_n - \frac{\partial P}{\partial x_n})$$

is

$$(x_1 + \frac{\partial Q}{\partial x_1}, \ldots, x_n + \frac{\partial Q}{\partial x_n}),$$

where

$$Q = \sum_{T \in \mathbb{T}} \frac{1}{|\text{Aut } T|} w_P(T),$$

$\mathbb{T}$ is the set of isomorphism classes of unrooted trees,

$$w_P(T) = \sum_{l: E(T) \rightarrow \{1, \ldots, n\}} \prod_{v \in V(T)} \frac{\partial^s}{\partial x_{l(e_1)} \cdots \partial x_{l(e_1)}} P,$$

and $\{e_1, \ldots, e_s\}$ is the set of edges adjacent to $v$. 

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Zhao’s Formula and the Gap Theorem


**Zhao’s Formula:** Let $P \in \mathbb{C}[x_1, \ldots, x_n]$ be homogeneous. For each $m \leq 1$ let $\mathbb{T}_m$ denote the set of unrooted tree isomorphism classes with $m$ vertices and set

$$Q^{(m)} = \sum_{T \in \mathbb{T}_m} \frac{1}{|\text{Aut } T|} w_P(T).$$

Then $Q^{(1)} = P$ and for $m \geq 2$,

$$Q^{(m)} = \frac{1}{2(m-1)} \sum_{k,l \geq 1, k+l=m} (\nabla Q^{(k)} \cdot \nabla Q^{(l)}).$$

**Gap Theorem:** If

$$Q^{(M+1)} = Q^{(M+2)} = \cdots = Q^{(2M)} = 0$$

then $Q^{(k)} = 0$ for all $k > M$. 
**Grossman-Larson-Wright Tree Module**

The Grossman-Larson rooted algebra $\mathcal{H}$ is the set of finite rational linear combinations of *rooted* trees. A computation in $\mathcal{H}$:

\[
\begin{array}{c}
\text{trees} \\
\times \\
= \\
\text{trees}
\end{array}
\]

The Grossman-Larson-Wright tree module $\mathcal{M}$ is the set of finite rational linear combinations of *unrooted* trees. A computation in $\mathcal{M}$:

\[
\begin{array}{c}
\text{trees} \\
\times \\
= 2 \\
\text{trees} \\
+ 2
\end{array}
\]
Special Classes of Trees

**Definition:** $V(e)$ denotes the set of all $T \in T$ which contain at least one vertex of degree $> e$.

**Example:** $\in V(1), V(2)$

**Definition:** $C(r)$ denotes the set of all $T \in T$ which contain a naked $r$-chain.

**Example:** $\in C(2), C(3)$
A Nice Property of $w_P$


Extend $w_P$ to a linear mapping $W_P : \mathcal{M} \to \mathbb{C}[x_1, \ldots, x_n]$.

Let $\mathcal{N}(r, e)$ denote the $\mathcal{H}$-submodule of $\mathcal{M}$ generated by $V(e)$ and $C(r)$.

**Theorem:** If $P \in \mathbb{C}[x_1, \ldots, x_n]$ is homogeneous of degree $e \geq 2$ and satisfies

$$
\left( \frac{\partial^2 P}{\partial x_i \partial x_j} \right)^r = 0
$$

for some $r \geq 1$, then

$$W_P(\mathcal{N}(r, e)) = 0.$$
A Combinatorial Problem

**Question:** For which \((m, r, e)\) do we have
\[ T_m \subseteq N(r, e)? \]

**Notation:** Let \( T_m(r, e) = T_m \setminus (V(e) \cup C(r)) \).

**Notation:** Given \( S \in T_{rt} \) and \( T \in C(r) \), let \([S \times T]_{r,e}\) denote the expression obtained from \( S \times T \) by dropping those terms from \( S \times T \) which belong to \( V(e) \cup C(r) \).

**Equivalent Question:** For which \((m, r, e)\) do we have
\[ T_m(r, e) \subseteq \text{span}_Q\{[S \times T]_{r,e} : S \in T_{rt}, T \in C(r)\}? \]
For example,

\[
\begin{align*}
\begin{array}{cccccc}
\times & = & 2 & +2 & +6 & +2 & +2 & +2 & +2 \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{cccccc}
\left[ \begin{array}{c}
\times \\
\end{array} \right]_{4,3} & = & 6 & +2 & +2 \\
\end{array}
\end{align*}
\]

Do the latter expressions generate all trees with 7 vertices, no naked 4-chains, and all degrees \( \leq 3 \)?
Coefficient Matrix for $m = 4, r = 3, e = 3$

\[
\begin{bmatrix}
2 & 1 \\
1 & 0
\end{bmatrix}
\]

Hence $T_4(3, 3) \subseteq N(3, 3)$

In general, when does coefficient matrix have full rank?
The Digraph Associated with a Matrix

Example:

\[
A = \begin{bmatrix}
1 & 0 & 2 & 0 \\
0 & 3 & 4 & 0 \\
0 & 0 & 5 & 6 \\
7 & 8 & 0 & 9 \\
0 & 0 & 10 & 0 \\
0 & 0 & 11 & 12
\end{bmatrix}
\]

\[
G_A =
\]

\((v_i, k, v_j) \in G_A \iff a_{ij}a_{ik} \neq 0\)
Row Subset Selection

Row Subset Selection Digraph: Given an $m \times n$ matrix $A$, for each $j \leq n$ let $R_j = \{i \leq m : a_{ij} \neq 0\}$. Let $\emptyset \neq R(v_j) \subseteq R_j$ for each $j \leq n$. Restrict outgoing edges of $v_j$ to edge labels in $R(v_j)$.

Example:

$R(v_1) = \{1\}, R(v_2) = \{2\}, R(v_3) = \{5\}, R(v_4) = \{3, 4\},$

$G_R =$

\[1 \rightarrow v_1 \rightarrow 1 \leftarrow 4 \rightarrow v_2 \rightarrow 2 \leftarrow 4 \rightarrow v_4 \rightarrow 3,4 \leftarrow 3 \rightarrow v_3 \rightarrow 5 \leftarrow v_1\]
Sufficient Conditions for Full Rank

**Theorem:** Let $A = (a_{ij})$ be a $m \times n$ matrix over the reals, let

$$R : \{v_1, \ldots, v_n\} \rightarrow 2^{\{1,\ldots,m\}}$$

be a row-selection function which satisfies

$$\emptyset \neq R(v_j) \subseteq R_j$$

for all $k \leq n$, and let $G_R$ be the row selection subgraph of $G_A$ defined by $R$.

1. If $G_R$ has no directed cycles of length $\geq 2$ then the rows chosen by $R$ span $\mathbb{R}^n$.

2. If $A$ is non-negative and $G_R$ has no directed cycles of even length then the rows chosen by $R$ span $\mathbb{R}^n$.

**Proof:** Choose one row per column arbitrarily using the row subset selection function $R$. No cycles of length 2 guarantee that $n$ distinct rows are chosen. The other hypotheses guarantee that the determinant of the corresponding $n \times n$ submatrix of $A$ is non-zero.
Connection to Spanning Sets

Assume \( \{x_1, x_2, x_3, x_4\} \) spans \( V \). Let

\[
\begin{align*}
y_1 &= x_1 + 2x_3 \\
y_2 &= 3x_2 + 4x_3 \\
y_3 &= 5x_3 + 6x_4 \\
y_4 &= 7x_1 + 8x_2 + 9x_4 \\
y_5 &= 10x_3 \\
y_6 &= 11x_3 + 12x_4
\end{align*}
\]

Form \( A, G_A, \) and \( G_R \) as before.

Row indices and edge labels are \( y_1, \ldots, y_6 \)

Column indices and vertex labels are \( x_1, \ldots, x_4 \)

\( G_R \) has no non-trivial cycles.

Therefore \( \{y_1, y_2, y_3, y_4, y_5, y_6\} \) spans \( V \).
**Theorem:** \( \mathcal{T}_m \subseteq \mathcal{N}(3, \infty) \) for \( m \geq 3 \).

**Proof:** \( \mathcal{T}_3 = \{\} \subseteq \mathcal{C}(3) \subseteq \mathcal{N}(3, \infty) \).

For \( m \geq 4 \), each \( T \in \mathbb{T}_m(3, \infty) \) falls into one of two disjoint categories:

**Category I:** \( T = \)

\[
\text{height}(X) = \text{diameter}(T) - 2, \quad p > 1
\]

**Category II:** \( T = \)

\[
\text{height}(X_1) = \text{diameter}(T) - 3, \quad j \geq 2, \quad |V(X_1)| \text{ maximal}
\]
Examples

Falls into Category I in three ways

Falls into Category II in two ways
$G_R$ Construction

$R(\text{tree}) = \{ [\cdot \cdot \cdot \cdot \cdot \cdot], [\cdot \cdot \cdot \cdot \cdot \cdot] \}$

other trees

[Diagram of tree structures]
Complete description of $G_R$

Edges to larger diameter trees not depicted:

$G_R$ has no non-trivial directed cycles, hence $\mathbb{T}_m(3, \infty)$ is spanned by all $\ldots$ and $\ldots$. 
Theorem: $T_8 \subseteq \mathcal{N}(4, 4)$.

Proof: The trees in $T_8(4, 4)$ are
$R(D_5) = [\bullet \cdot], \quad R(D_6) = [\checkmark \cdot], \quad R(D_7) = [\bullet \cdot],$

$R(D_8) = [\checkmark \cdot], \quad R(D_{10}) = [\bullet \cdot], \quad R(D_{11}) = [\bullet \cdot],$

$R(D_{12}) = [\checkmark \cdot], \quad R(D_{13}) = [\checkmark \cdot], \quad R(D_{14}) = [\checkmark \cdot],$

$R(D_{15}) = [\checkmark \cdot], \quad R(D_{18}) = [\bullet \cdot], \quad R(D_{19}) = [\checkmark \cdot],$

$R(D_{20}) = [\checkmark \cdot], \quad R(D_{22}) = [\checkmark \cdot].$

$G_R$ has exactly one non-trivial odd cycle:

$$D_6 \to D_8 \to D_{13} \to D_6.$$
Further Research

For which other values of $r$, $m$, and $e$ can we prove $\mathcal{T}_m \subseteq \mathcal{N}(r, e)$?

Can we avoid odd cycles by considering large $m$?

Can we exploit other properties of trees besides diameter?

We have $\mathcal{N}(r + 1, e) \subseteq \mathcal{N}(r, e)$. Can we descend from $\mathcal{T}_m \subseteq \mathcal{N}(r, e)$ to $\mathcal{T}_m \subseteq \mathcal{N}(r + 1, e)$?

Are there other combinatorial problems that are solvable using this method?