Kuratowski’s Theorem

**Kuratowski subgraph of a graph:** A subgraph which can be described as subdivision of $K_5$ or $K_{3,3}$ (interrupt edges by degree 2 vertices).

**Petersen Graph:** Satisfies $e \leq 3v - 6$ but not $(k - 2)e \leq kv - 6$ using $k = 5$, hence non-planar. Circle-chord method yields a $K_{33}$ configuration: see illustration.

**Branch and subdivision vertices in a Kuratowski subgraph:** Branch vertices are the original vertices of $K_5$ and $K_{33}$. Subdivision vertices are the inserted vertices of degree 2.

**Minimal non-planar graph:** A non-planar graph such that every proper subgraph is planar.

**X-lobe of a graph $G$:** Let $X$ be subset of vertices of $G$ and let $G_i$ be a connected component of $G - X$. An $X$-lobe of $G$ is the induced subgraph $G[G_i + X]$.

**Edge contraction:** Let $G$ be a graph and let $e = \{x, y\}$ be an edge in $G$. $G \cdot e$ is the graph obtained from $G$ by shrinking the edge $xy$ down to a point $z$. In the process we lose the vertices $x$ and $y$, gain the vertex $(xy)$, and any edge $xv$ or $yv$ in $G$ becomes the edge $(xy)v$ in $G \cdot e$.

**Example:** The Petersen graph can be contracted down to $K_5$. See illustration. If $G$ is planar then $G \cdot e$ is planar. So if some contraction of $G$ is nonplanar then $G$ is nonplanar. Hence $P$ is nonplanar.

**Lemma 1:** Let $C$ be a cycle of a planar graph $G$. Then there is a way to draw $G$ so that the edges of $C$ all border the infinite region.

**Proof:** Stuff everything inside as before.

**Lemma 2:** Every minimal non-planar graph $G$ (all proper subgraphs planar) is 2-connected.

**Proof:** First note that any minimal non-planar graph must be connected. We must show that there are no cut-vertices.

Suppose $x$ is a cut vertex. Let $G - x$ have components $H_1, \ldots, H_k$. Each lobe $G[H_i + x]$ is planar by minimality of $G$. Each non-tree among these can be redrawn so that $x$ is bordering the infinite region. Each tree among them has $x$ bordering the infinite region. These can be glued together to form a planar representation of $G$. Contradiction. So there are no cut vertices.
Lemma 3: Suppose $G - x - y$ is not connected. If $G$ is non-planar then adding the edge $xy$ to some $\{x, y\}$-lobe of $G$ yields a non-planar graph.

Proof: Let the components of $G - x - y$ be $H_1, \ldots, H_k$. Suppose every $G[H_i + x + y] + xy$ is planar. Draw each such configuration so that the edge $xy$ borders the infinite region. There is a way to glue all these things together to create a planar graph, and this includes $G$ as a subgraph. Contradiction. So some $G[H_i + x + y] + xy$ is non-planar.

Lemma 4: If there exists a minimal example $G$ of a non-planar graph with no Kuratowski subgraph, then it is 3-connected.

Proof: $G$ is a minimal non-planar graph. By Lemma 2 is $G$ is 2-connected. If $G$ is not 3-connected then it has a minimal vertex cut $\{x, y\}$. Let the connected components of $G - x - y$ be $H_1, \ldots, H_k$. Suppose every $G[H_i + x + y] + xy$ is planar. Draw each such configuration so that the edge $xy$ borders the infinite region. There is a way to glue all these things together to create a planar graph, and this includes $G$ as a subgraph. Contradiction. So some $G[H_i + x + y] + xy$ is non-planar.

Lemma 5: Every 3-connected graph $G$ with at least 5 vertices has an edge $e$ such that $G \cdot e$ is also 3-connected.

Proof: Suppose this edge cannot be found. Let $G$ be 3-connected and let $e = xy$ be an edge in $G$. Since $G \cdot e$ is not 3-connected, it has a vertex cut $\{u, v\}$. We claim that $(xy) = u$ or $(xy) = v$.

Suppose in fact $(xy) \neq u$ and $(xy) \neq v$. Then there must be some vertex in $(G \cdot e) - u - v$ with no path to $(xy)$. Since $G$ has $v \geq 5$ vertices, there must be some vertex in $G - u - v$ with no path to $x$ and no path to $y$. Contradiction.

So now we know that $G \cdot e$ has a separating set of the form $\{(xy), z\}$. This creates a separating set $\{x, y, z\}$. Of all ways to choose the edge $e = xy$, choose one which maximizes the vertices in the largest connected component of $G - x - y - z$. Since $\{x, y, z\}$ is a minimal vertex cut of $G$, $x$, $y$ and $z$ have edges to each component of $G - x - y - z$. Now let $H$ be the largest component of $G - x - y - z$ and let $H'$ be another component of $G - x - y - z$. Let $u$ be a neighbor of $z$ in $H'$. Let $v$ be such that $G$ has a separating set
\{z, u, v\}. To achieve a contradiction we will find a connected component of \(G - z - u - v\) that is larger than \(H\).

First note that \(G[H + x + y - v]\) is connected: Consider the cases.

Case 1: \(v = x\). Then \(G[H + y]\) is connected.

Case 2: \(v = y\). Then \(G[H + x]\) is connected.

Case 3: \(v \in H\). We know that \(G - z - v\) is connected. Given two vertices in \(G[H + x + y - v]\), find a path between them in \(G - z - v\) and shrink it to a path in \(G[H + x + y - v]\).

Now that we know that \(G[H + x + y - v]\) is connected, it has to belong to a connected component \(H''\) of \(G - z - u - v\) which has at least as many vertices as \(G[H + x + y - v]\) and strictly greater vertices than \(H\). Contradiction. So yes, we can find \(e \in G\) such that \(G \cdot e\) is 3-connected.

**Lemma 6:** If \(G \cdot e\) has a Kuratowski subgraph then so does \(G\).

**Proof:** Let \(K\) be a Kuratowski subgraph in \(G \cdot e\). Write \(e = xy\). If \((xy) \notin K\) then \(K\) is a Kuratowski subgraph of \(G\). Now suppose \((xy) \in K\). If \((xy)\) is a subdivision vertex of \(K\), let the edges it belongs to be \(u(xy)\) and \((xy)v\).

By considering the possibilities in \(G\) we can see that \(G\) has a Kuratowski subgraph. If \((xy)\) is a branch vertex of \(K\) and exactly one of the edges \((xy)u_i\) in \(K\) corresponds to \(xu_i\) in \(G\) and the rest correspond to \(yu_i\) in \(G\) then \(x\) is a subdivision vertex of a Kuratowski subgraph in \(G\) (or \(y\) if the roles of \(x\) and \(y\) are reversed). The only remaining case is when \(K\) is a subdivision of \(K_5\) and the four edges \((xy)u_1, (xy)u_2, (xy)u_3, (xy)u_4\) in \(K\) correspond to \(xu_1, xu_2, yu_3, yu_4\) in \(G\). Writing the branch vertices of \(K\) as \((xy), v_1, v_2, v_3, v_4\), there are paths joining each \(v_i\) to \(v_j\) as well as paths from \(x\) to \(v_1\) and \(v_2\) and paths from \(y\) to \(v_3\) and \(v_3\), as well as the edge \(xy\). Tossing the \(v_1v_2\) path and the \(v_3v_4\) path, we obtain a subdivision of \(K_{33}\) out of the remaining paths, with branch vertices \(x, v_3, v_4\) on the right and branch vertices \(y, v_1, v_2\) on the left.

**Lemma 7:** If \(G\) does not have a Kuratowski subgraph and \(G \cdot e\) is 3-connected and planar, then \(G\) is planar.

**Proof:** We know by Lemma 6 that \(G \cdot e\) does not have a Kuratowski subgraph. Now draw a planar representation of \(G \cdot e\). Removing \((xy)\), the remaining graph is 2-connected. Therefore \((xy)\) and the edges to its neighbors in \(G \cdot e\) are bounded by a cycle \(C\). The vertices of \(C\) belong to \(G\). The neighbors
of $x$ and $y$ in $G$ belong to $C$. Let the neighbors of $x$ be $x_1, \ldots, x_j$ in cyclic order around $C$ and let the neighbors of $y$ be $y_1, \ldots, y_k$ in cyclic order around $C$. Note that there could be some overlap among these sets of neighbors. It is clear that $G - x$ and $G - y$ are planar since they are isomorphic to subgraphs of $G \cdot e$, and if $x$ has $\leq 1$ neighbors then we can insert $y$ and its edges to $x, y_1, \ldots, y_k$ to create a planar representation of $G$. Now assume that $x$ has at least 2 neighbors in $C$. We will consider the ways $y_1, \ldots, y_k$ can be distributed around $C$.

Case 1: $y$ has at least three neighbors $z_1, z_2, z_3$ in common with $x$. Using $C$ we can create a $K_5$ subdivision in $G$. Contradiction. So Case 1 cannot happen.

Case 2: $y$ shares at most two neighbors in common with $x$ and the rest of neighbors of $y$ all fall between two consecutive neighbors $x_i, x_{i+1}$ of $x$. Then we can insert $y$ in the triangle formed by $x, x_i, x_{i+1}$ and draw all the edges out of $y$ to create a planar representation of $G$.

Case 3: $y$ shares at most two neighbors in common with $x$ but the rest of the neighbors of $y$ do not fall between two consecutive neighbors $x_i, x_{i+1}$ of $x$. In other words, $y$ has neighbors $z_1$ and $z_2$ that alternate with neighbors $x_i$ and $x_{i+1}$ of $x$. Using $C$ we can create a $K_{3,3}$ subdivision in $G$. Contradiction. So Case 3 cannot happen.

**Theorem:** Every graph that does not have a Kuratowski subgraph is planar.

**Proof:** If the theorem is false, then there is a minimal counterexample, $G$. $G$ is non-planar, does not have a Kuratowski subgraph, and by Lemma 4 $G$ is 3-connected. Since $K_4$ and its subgraphs are planar, $G$ must have at least 5 vertices. By Lemma 5, $G$ has an edge $e$ such that $G \cdot e$ is 3-connected. By Lemma 6, $G \cdot e$ does not have a Kuratowski subgraph. By minimality of $G$, $G \cdot e$ must be planar. By Lemma 7, $G$ must be planar. Contradiction. So the theorem is true.