Course Proposal

Title: Math 492 Mathematics Capstone Experience

Topic: Enumerative Combinatorics

Instructor: Dan Singer

Course Description: Enumerative combinatorics is the art of counting. To count discrete objects (permutations, combinations, integer partitions, trees, graphs etc) is to understand their structure down to the last detail. Topics will include elementary counting techniques, bijections, recurrence relations, generating functions, combinatorial species, group actions, and Pólya’s method for counting objects with symmetry. Students will use these techniques to make conjectures, prove theorems, read research papers, and communicate mathematical ideas in both written and oral form. They will see connections among the following mathematical disciplines: enumerative combinatorics, abstract algebra, real analysis, and number theory.

Prerequisites: Two of the following: Math 316, Math 345, Math 375.

Instructional Materials: Lecture notes, handouts, research papers. I will not be using a textbook, but will consult the books listed in Appendix A. The research papers available for students to choose from are listed in Appendix B.

Course Format: I will give lectures on counting techniques and work out a number of examples throughout the course. Students will be assigned homework on a weekly basis. At the beginning of the course I will hand out a list of research papers and list of projects. Each student must pick a research paper to read and a project to work on. Both the research paper and the project must be presented to the class in the form of an oral presentation. The project must also be submitted in written form. The research paper must be chosen by week 5 and the research project must be chosen by week 10. Students have the option of working alone or in groups of two on all assignments.

Grading Policy: All assignments are awarded points on a 0 to 100 scale. 90-100 points for an A, 80-89 points for a B, 70-79 points for a C, 60-69 points for a D, 0-59 points for an F. Assignments will consist of weekly homework problems, a research paper to be digested and presented to the class, and a project to be presented to the class and also submitted in written form. The final grade will be calculated as follows: 30% homework score average, 35% research paper, 35% project, letter grade to be awarded according to how this weighted average falls on the grade scale indicated above.

Syllabus:

Weeks 1 and 2: addition principle, multiplication principle, inclusion-exclusion

Weeks 3 and 4: bijections, combinatorial arguments for proving binomial coefficient identities, combinatorial arguments for proving fibonacci number identities

Week 5 and 6: partitions and $q$-series identities

Week 7: recurrence relations
Week 8: generating functions and the Lagrange inversion formula

Weeks 9 and 10: combinatorial species

Weeks 11 and 12: group actions and Pólya theory

Weeks 13, 14, and 15: Reserved for student presentations of research papers and projects.

Projects for a capstone course on enumerative combinatorics:

1. Find the number of $n \times n$ chess boards in which exactly two squares in every row and column are occupied by pawns and the remaining squares are empty.

2. Find combinatorial proofs of binomial coefficient identities found in Riordan’s book – as many as possible.

3. Find q-analogues of binomial coefficient identities found in Riordan’s book – as many as possible.

4. (Exercise 79 in Andrews.) Prove that the number of partitions of $n$ in which no part appears exactly once equals the number of partitions into parts not congruent to ±1 (mod 6).

5. (Exercise 102 in Andrews.) show that the generating function for self-conjugate partitions with each part $\leq N$ is

$$
\sum_{j=0}^{N} q^{j^2} \left[\begin{array}{c} N \\ j \end{array}\right] q^j
$$

6. (Exercise 5.6 in Stanley, Volume II.) Let $\chi(K_{mn}, q)$ denote the chromatic polynomial of the complete bipartite graph $K_{mn}$. Show that

$$
\sum_{m,n \geq 0} \chi(K_{mn}, q) \frac{x^m y^n}{m! n!} = (e^x + e^y - 1)^q.
$$

7. (Exercise 5.7 in Stanley, Volume II.) Using combinatorial species applied to up-down permutations, give a combinatorial proof of the identities

$$
1 + \tan^2 x = \sec^2 x
$$

and

$$
\tan (x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}.
$$

8. (Exercise 5.9 in Stanley, Volume II.) Let $S$ be a “structure” that can be put on a finite set $S$ by choosing a partition of $S$ and putting a “connected” structure on each block, so that the exponential formula is applicable. Let $f(n)$ be the number of structures that can be put on an $n$-element set, and let $F(x) = E_f(x)$, the exponential generating function of $f$. 

2
(a) Let \( g(n) \) be the number of structures that can be put on an \( n \)-set so that every connected component has even cardinality. Show that
\[ E_g(x) = \sqrt{F(x)F(-x)}. \]

(b) Let \( e(n) \) be the number of structures that can be put on an \( n \)-set so that the number of connected components is even. Show that
\[ E_e(x) = \frac{1}{2} \left( F(x) + \frac{1}{F(x)} \right). \]

9. (Exercise 5.11 in Stanley, Volume II.)
(a) Let \( a(n) \) be the number of permutations in \( S_n \) which have a square root. Show that
\[ \sum_{n \geq 0} a(n) \frac{x^n}{n!} = \left( \frac{1 + x}{1 - x} \right)^{1/2} \prod_{k \geq 1} \cosh \frac{x^{2k}}{2k}. \]

(b) Deduce from (a) that \( a(2n+1) = (2n+1)a(2n) \). Is there a simple combinatorial proof?

10. (Exercise 5.28 in Stanley, Volume II.) Let \( k \) be a positive integer. A \( k \)-edge colored tree is a tree whose edges are colored from a set of \( k \) colors such that any two edges with a common vertex have different colors. Show that the number \( T_k(n) \) of \( k \)-edge colored trees on the vertex set \([n]\) is given by
\[ T_k(n) = k(nk - n)(nk - n - 1) \cdots (nk - 2n + 3) = k(n-2)! \binom{nk-3}{n-2}. \]

11. (Exercise 14.45 in Brualdi.) A stained glass window in the form of a 4-by-4 chessboard has 16 squares, each of which is colored red or blue (the colors are transparent and the window can be looked at from the other side). Determine the generating function for the number of different stained glass windows and the total number of stained glass windows.

12. (Exercise 14.49 in Brualdi.) Ten balls are stacked in a triangular array with 1 atop 2 atop 3 atop 4. (Think of billiards.) The triangular array is free to rotate. Find the generating function for the number of nonequivalent colorings with the colors red and blue. Find the generating function if we are also allowed to turn over the array.

Appendix A: Source Materials for Lecture Notes


**Appendix B: Research Papers**


