Questions for Section 15: Factor-Group Computations and Simple Groups

1. This is a needlessly long and complicated section. We will simplify it. The thing to concentrate on is Theorem 15.8 (page 147), Theorem 15.9 (page 147), Theorem 15.15 (page 149), Theorem 15.16 (page 149), Theorem 15.18 (page 150). We will skip Theorem 15.20 (page 150) because it relies on Section 7, which we skipped. Let’s chase down the other theorems.

2. Let $H$ and $K$ be groups. Then $H \times K/\{(h, e) : h \in H\} \cong H$. Proof: Verify that $\phi : H \times K \to H$ defined by $\phi(h, k) = h$ is a group homomorphism with kernel $\{(h, e) : h \in H\}$. So for example $\mathbb{Z}_m \times \mathbb{Z}_n/\langle(1, 0)\rangle \cong \mathbb{Z}_m$.

3. A factor group of a cyclic group is a cycle. Proof: Let $G$ be a group and $N$ a subgroup of $G$. Assume $G = \langle a \rangle$. Then $G/N = \langle aN \rangle$. For example, if $G = \mathbb{Z}_n$ and $N = \langle s \rangle$ then $G/N = \langle 1 + \langle s \rangle \rangle$. Verify this for $n = 12$ and $s = 3$.

4. $A_5$ is a simple group, meaning that if $N \leq A_n$ is normal then $N = \{e\}$ or $N = A_n$. The significance of this result is that it implies that there is no general formula for the quintic equation $ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$ by extracting roots (as there is, for example, for the equation $ax^2 + bx + c = 0$, where $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$). Proof: Let $N$ be a normal subgroup of $A_5$, and assume that $N \neq \{e\}$. We will prove that $N = A_5$. Let $\sigma \in N\{e\}$ be given, and assume $\sigma = c_1c_2 \cdots c_k$ is the factorization of $\sigma$ into disjoint cycles of length $\geq 1$ each (omitting 1-cycles for simplicity). Consider the following cases:

Case 1. $k = 1$. Then we must have $\sigma = (a, b, c)$ or $\sigma = (a, b, c, d, e)$. In the latter case, $(a, e, d, c, b) \in N$ and $(b, d, c)(a, b, c, d, e)(b, c, d) = (a, d, b, c, e) \in N$, therefore $(a, e, d, c, b)(a, d, b, c, e) = (a, c, d) \in N$. In either case, $N$ contains a 3-cycle.

Case 2. $k = 2$. Then we must have $\sigma = (a, b)(c, d)$. Therefore $(c, f, d)(a, b)(c, d)(c, d, f) = (a, b)(c, f) \in N$, therefore $(a, b)(c, d)(a, b)(c, f) = (c, f, d) \in N$. Hence $N$ contains a 3-cycle.

So we know that in all cases $N$ contains a 3-cycle. Call it $(x, y, z)$. Then $N$ also contains $(x, z, y)$. So $N$ contains every 3-cycle which involves the 3 numbers $x$, $y$, and $z$. Now we will show that $N$ contains every 3-cycle bar
none. All we need to do is show that if \( N \) contains a 3-cycle then it contains another one with one number in the cycle replaced by another number not in the cycle. This is true: if \( (a, b, c) \in N \) and then \( (c, x)(a, b)(a, b, c)(a, b)(c, x) = (a, x, b) \in N \).

Now we will show that every product of two 2-cycles belongs to \( N \). We have \( (a, b)(a, b) = e \in N \), \( (a, b)(a, c) = (a, c, b) \in N \), and \( (a, b)(c, d) = (a, b, c)(b, c, d) \in N \).

Now we will show that \( N = A_5 \). We already have \( N \subseteq A_5 \). Let \( \sigma \in A_5 \). Then \( \sigma = \tau_1 \tau_2 \cdots \tau_{2k} \) for some choice of 2-cycles \( \tau_1, \ldots, \tau_{2k} \). But we have just proved that \( \tau_1 \tau_2 \in N \), \( \tau_3 \tau_4 \in N \), \( \ldots \), \( \tau_{2k-1} \tau_{2k} \in N \). Therefore \( \sigma \in N \). Therefore \( A_5 \subseteq N \). Therefore \( N = A_5 \).

5. Let \( \phi : G \to G' \) be a group homomorphism. If \( N \leq G \) is normal then \( \phi(N) \leq G' \) is normal. If \( N' \leq G' \) is normal then \( \phi^{-1}(N') \leq G \) is normal.  
Proof: Exercises 35 and 36.

6. Let \( N \leq G \) be normal. We say that \( N \) is maximal normal in case there is no subgroup \( N' \leq G \) such that \( N \subseteq N' \) and \( N' \neq N \) and \( N' \neq G \). Theorem: \( N \) is maximal normal if and only if \( G/N \) is simple. Proof: Assume \( N \) is maximal normal. Let \( H \leq G/N \) be normal. We must show that \( H = \{eN\} \) or \( H = G/N \). Let \( X = \{a \in G : aN \in H\} \). Then a simple argument shows that \( H \) is a normal subgroup of \( G \) and \( N \subseteq H \). Therefore \( H = N \) (in which case \( X = \{e\} \) or \( H = G \) (in which case \( X = G/N \)). Therefore \( G/N \) is simple. Conversely, assume \( G/N \) is simple. Let \( N' \leq G \) be normal such that \( N \subseteq N' \). Then a simple argument shows that \( Y = \{aN : a \in N'\} \) is a normal subgroup of \( G/N \). Therefore \( Y = \{e\} \), which implies \( aN = eN \) for all \( a \in N' \), which implies \( a \in N \) for all \( a \in N' \), which implies \( N' \subseteq N \), which implies \( N' = N \), or \( Y = G/N \), which implies for each \( g \in G \) there exists \( a \in N' \) such that \( gN = aN \subseteq N' \), which implies \( g \in N' \) for all \( g \in G \), which implies \( N' = G \). Hence \( N \) is maximal normal.

**Homework for Section 14, due ?? (only the starred problems will be graded):**

5*, 25*, 27*, 35*, 36*

Hints:

5. Compute all the left cosets. Then find an isomorphism between the coset group and \( \mathbb{Z}_4 \times \mathbb{Z}_8 \). Alternatively, find a homomorphism \( \phi : \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_8 \to \mathbb{Z}_4 \times \mathbb{Z}_8 \) which is onto and which has kernel \( \langle (1, 2, 4) \rangle \) and use Theorem 14.11.
25. \( U \) is the group of complex numbers \( z \) such that \( |z| = 1 \). The group operation is multiplication. Prove that the function \( \phi : U \rightarrow U \) defined by \( \phi(z) = z^2 \) is an onto group homomorphism with kernel \( \langle -1 \rangle \) and use Theorem 14.11.

27. Prove that the function \( \phi : \mathbb{R} \rightarrow U \) defined by \( \phi(\theta) = e^{2\pi i \theta} \) is an onto group homomorphism with kernel \( \mathbb{Z} \) and use Theorem 14.11.