Give complete proofs for all problems.

1. Let $\phi : \mathbb{C} \to M_2(\mathbb{R})$ be defined by

$$\phi(a + bi) = \begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

Is $\phi$ a ring homomorphism? Justify your answer.

2. Let

$$R = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}.$$

Prove that $R$ is a field. (You do not have to check that addition is associative or that multiplication is distributive, but you must prove the other properties of a field.)

3. State Euler’s Theorem correctly, then use it to find the remainder of $65^{5906}$ upon division by 18, verifying that the hypotheses of Euler’s Theorem are met.

4. Find all roots of $x^4 + 3$ in $\mathbb{Z}_7$, then factor $x^4 + 3$ into irreducible factors in $\mathbb{Z}_7[x]$. Explain why your factors are irreducible.

5. Factor the polynomial $5x^5 + 2x^4 + 7x^3 + 7x^2 + 14x - 35$ into irreducible factors in $\mathbb{Q}[x]$. Hint: 1 is a root. Use Eisenstein’s Criterion to prove irreducibility of one of the factors.