Questions for Section 35: Pointwise and Uniform Convergence

1. One reason for working with power series $\sum a_n x^n$ is to find solutions to differential equations that cannot be expressed in terms of known functions like $e^x$ or $\sin x$. For example: find a function $f : (-\alpha, \alpha) \to \mathbb{R}$ for which $y = f(x)$ is a solution to the differential equation $xy'' + y' + xy = 0$. It turns out that the solution is the Bessel function

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n (n!)^2} x^{2n} = 1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 - \frac{1}{2304}x^6 + \cdots,$$

which has radius of convergence $R = +\infty$.

2. Another reason for working with power series is to integrate functions which do not have simple antiderivatives. For example, what is $\int_{0}^{\pi/2} \sin(x^2) \, dx$? If we were not concerned with rigor (which of course we are in this course!), we could do this:

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1},$$

$$\sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+2},$$

$$\int_{0}^{\pi/2} \sin(x^2) \, dx = \int_{0}^{\pi/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+2} \, dx = \sum_{n=0}^{\infty} \int_{0}^{\pi/2} \frac{(-1)^n}{(2n+1)!} x^{4n+2} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left( \frac{\pi}{2} \right)^{4n+3}. $$

3. The unrigorous part of the previous calculation was the assertion that

$$\int_{0}^{\pi/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+2} \, dx = \sum_{n=0}^{\infty} \int_{0}^{\pi/2} \frac{(-1)^n}{(2n+1)!} x^{4n+2} \, dx.$$

To see this, for each integer $n \geq 0$ define $f_n : [0, \pi/2] \to \mathbb{R}$ by

$$f_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{4k+2}. $$
The equation above can be re-written as
\[
\int_0^{\pi/2} \left( \lim_{n \to \infty} f_n(x) \right) \, dx = \lim_{n \to \infty} \left( \int_0^{\pi/2} f_n(x) \, dx \right).
\]

However, in Example 35.3, pp. 321–322, we see that we cannot always exchange the limit operation with integration (which is another limit operation) and get the same answer. Therefore, without recourse to some sort of theorem about the interchangeability of limit operations, the steps in Comment 2 above are not justified.

4. We are interested in studying properties of a function \( f : [a, b] \to \mathbb{R} \) such that \( f(x) = \lim_{n \to \infty} f_n(x) \), where \( f_n : [a, b] \to \mathbb{R} \) is a function for each \( n \in \mathbb{N} \). All power series have this form, as we can see in Comment 3 above. We can also take \( f_n(x) \) to be the \( n^{th} \) Taylor Polynomial of \( f \) at some \( x_0 \). Here are three questions we can ask about \( f \):

Question 1. If each \( f_n : [a, b] \to \mathbb{R} \) is continuous at \( c \), is \( f : [a, b] \to \mathbb{R} \) continuous at \( c \)? This is equivalent to asking if
\[
\lim_{x \to c} f(x) = f(c),
\]
which is equivalent to
\[
\lim_{x \to c} \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} f_n(c).
\]

Example 35.2, pp. 320–321 shows that this will not always be true.

Question 2. If each \( f_n : [a, b] \to \mathbb{R} \) is differentiable at \( c \), is \( f : [a, b] \to \mathbb{R} \) differentiable at \( c \)? In particular, is it safe to say that
\[
f'(c) = \lim_{n \to \infty} f'_n(c) ?
\]
This is equivalent to
\[
\lim_{x \to c} \lim_{n \to \infty} \frac{f_n(x) - f_n(c)}{x - c} = \lim_{n \to \infty} \lim_{x \to c} \frac{f_n(x) - f_n(c)}{x - c}.
\]

Example 35.4, page 322, shows what can go wrong.

Question 3. If each \( f_n : [a, b] \to \mathbb{R} \) is integrable on \([a, b]\), is \( f : [a, b] \to \mathbb{R} \) integrable on \([a, b]\)? Even if it is, can we say
\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} \int_a^b f_n(x) \, dx ?
\]
This is equivalent to
\[ \int_a^b \left( \lim_{n \to \infty} f_n(x) \right) \, dx = \lim_{n \to \infty} \left( \int_a^b f_n(x) \, dx \right). \]

Example 35.3, pp. 321–322 shows that we cannot take this for granted.

5. Let \( (f_n) \) be a sequence of functions which converges pointwise to a function \( f \) on the set \( S \). This means that \( f_n : S \to \mathbb{R} \) is a function for each \( n \in \mathbb{N} \) and that \( \lim f_n(x) = f(x) \) for each \( x \in S \). We say that \( (f_n) \) converges uniformly to \( f \) on \( S \) if for all \( \epsilon > 0 \) there exists \( N \) such that \( n > N \) implies \( |f_n(x) - f(x)| < \epsilon \) for every \( x \in S \). In other words, the same \( N \) works for every \( x \). There is a nice illustration of this in Figure 35.3, page 323. The function sequences in Examples 35.2 and 35.3 do not converge uniformly (proofs given in Examples 35.7 and 35.8), and these were the examples used in Comment 4 above which demonstrated that limit operations are not necessarily interchangeable. We will see in Section 36 that uniformly convergent function sequences \( (f_n) \) have the property that a variety of limit operations applied to them are interchangeable.

6. Let \( f_n : [0, 1] \to \mathbb{R} \) be defined by \( f_n(x) = \frac{x^n}{n} \) for each \( n \in \mathbb{N} \). Then \( \lim_{n \to \infty} f_n(x) = 0 \) for each \( x \in [0, 1] \). Let \( f : [0, 1] \to \mathbb{R} \) be defined by \( f(x) = 0 \) for all \( x \in [0, 1] \). We will prove that \( (f_n) \) converges to \( f \) uniformly on \([0, 1]\). Let \( \epsilon > 0 \) be given. We must find \( N \) such that \( n > N \) implies \( |f_n(x) - f(x)| < \epsilon \) for all \( x \in [0, 1] \). This is equivalent to \( |\frac{x^n}{n} - 0| < \epsilon \). Note that if \( x \in [0, 1] \) then \( |\frac{x^n}{n}| \leq \frac{1}{n} \), so we need \( n > \frac{1}{\epsilon} \). Hence we can use \( N = \frac{1}{\epsilon} \).

7. Instead of proving that \( (f_n) \) converges uniformly to \( f \) on \( S \) using the definition in Comment 5, we can use the following Cauchy Criterion for Function Sequences:

**Theorem:** Let \( (f_n) \) converge pointwise to \( f \) on \( S \). Then \( (f_n) \) converges uniformly to \( f \) if and only if for all \( \epsilon > 0 \) there exists \( N \) such that \( m, n > N \) implies \( |f_m(x) - f_n(x)| < \epsilon \) for all \( x \in S \).

**Proof:** Assume \( (f_n) \) converges uniformly to \( f \). Let \( \epsilon > 0 \) be given. By hypothesis, there exists \( N \) such that \( n > N \) implies \( |f_n(x) - f(x)| < \frac{\epsilon}{2} \) for all \( x \in S \). So if \( m, n > N \) then
\[
|f_m(x) - f_n(x)| = |f_m(x) - f(x) + f(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f(x) - f_n(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]
for all \( x \in S \). Hence \( (f_n) \) converges uniformly to \( f \) on \( S \).

Conversely, assume that \( (f_n) \) satisfies the Cauchy Criterion. Let \( \epsilon > 0 \) be given. Then there exists \( N \) such that \( m, n > N \) implies \( |f_m(x) - f_n(x)| < \frac{\epsilon}{2} \) for all \( x \in S \). Fix any \( n > N \) and any \( x \in S \). Since \( \lim_{m \to \infty} f_m(x) = f(x) \), there exists \( N' \) such that \( m > N' \) implies that \( |f_m(x) - f(x)| < \frac{\epsilon}{2} \). Let \( m_0 = 1 + \max(N, N') \). Then we have \( |f_{m_0}(x) - f_n(x)| < \frac{\epsilon}{2} \) and \( |f_{m_0}(x) - f(x)| < \frac{\epsilon}{2} \), therefore

\[
|f_n(x) - f(x)| = |f_n(x) - f_{m_0}(x) + f_{m_0}(x) - f(x)| \leq |f_{m_0}(x) - f_n(x)| + |f_{m_0}(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

Therefore \( (f_n) \) converges uniformly to \( f \) on \( S \).

Remark: Note that we can rephrase this theorem slightly to “\( (f_n) \) converges uniformly to a function on \( S \)” without having to mention \( f \). The reason for this is that if \( (f_n) \) satisfies the Cauchy Criterion cited above then for each \( x \in S \) the sequence \( (f_n(x)) \) is Cauchy, therefore converges to something which we can define as \( f(x) \).

8. An application of the Cauchy Criterion is the Weierstrass \( M \)-test, which is useful when dealing with power series.

**Weierstrass \( M \)-test**: Let \( (f_n) \) be a sequence of functions defined on \( S \). Suppose the function \( f_n \) is bounded by \( M_n \geq 0 \) for each \( n \in \mathbb{N} \) and that \( \sum M_n \) converges. Then \( f = \sum f_n \) converges uniformly on \( S \). In other words, if we define \( s_n = f_1 + \cdots + f_n \) for each \( n \in \mathbb{N} \) then \( (s_n) \) satisfies the Cauchy Criterion for Function Sequences, hence converges uniformly to some function \( f : S \to \mathbb{R} \).

**Proof**: Let \( \epsilon > 0 \) be given. We must show that there exists \( N \) such that \( n > m > N \) implies \( |s_n(x) - s_m(x)| < \epsilon \) for all \( x \in S \). Since \( \sum M_n \) converges, we can find \( N \) such that \( n > m > N \) implies

\[
|M_{m+1} + \cdots + M_n| < \epsilon
\]

by the Cauchy Criterion for Sequences. Therefore \( n > m > N \) implies

\[
|s_n(x) - s_m(x)| = |f_{m+1}(x) + \cdots + f_n(x)| \leq |f_{m+1}(x)| + \cdots + |f_n(x)| \leq M_{m+1} + \cdots + M_n < \epsilon.
\]
Hence \((s_n)\) satisfies the Cauchy Criterion for Function Sequences and must converge uniformly to some \(f : S \to \mathbb{R}\).

9. Let \(f_n : [0, \frac{\pi}{2}] \to \mathbb{R}\) be defined by \(f_n(x) = \frac{(-1)^n}{(2n+1)!}x^{4n+2}\) for each \(n \geq 0\). We will prove that \(\sum f_n\) converges uniformly to \(f : [0, \frac{\pi}{2}] \to \mathbb{R}\) be defined by \(f(x) = \sin(x^2)\). First note that \(|f_n(x)| \leq \frac{(\frac{\pi}{2})^{4n+2}}{(2n+1)!}\) for each \(n\). Applying the ratio test to the sequence \((\frac{(\frac{\pi}{2})^{4n+2}}{(2n+1)!})\) we obtain

\[
\lim \frac{(\frac{\pi}{2})^{4n+6}}{(2n+3)!} \cdot \frac{(\frac{\pi}{2})^{4n+2}}{(2n+1)!} = \left(\frac{\pi}{2}\right)^4 \lim \frac{1}{(2n+2)(2n+3)} = 0,
\]

hence \(\sum M_n\) converges, hence by the Weierstrass \(M\)-test \(\sum f_n\) converges to some \(f : [0, \frac{\pi}{2}] \to \mathbb{R}\). Looking at the proof of the Weierstrass \(M\)-test, the rule for \(f\) is

\[f(x) = \lim s_n(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{4n+2}.\]

We will prove that \(f(x) = \sin(x^2)\) for each \(x \in [0, \frac{\pi}{2}]\).

Note that \(\sin(x)\) and its first \(2n+1\) derivatives are all continuous on \([0, (\frac{\pi}{2})^2]\), therefore \(\sin(x)\) is approximated by its \(2n+1\)st Taylor polynomial

\[p_{2n+1}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!}x^{2k+1}\]

with error

\[|R_{2n+1}(x)| = \left|\frac{\sin^{(2n+2)}(c_x)}{(2n+2)!}x^{2n+2}\right| \leq \frac{(\frac{\pi}{2})^{4n+4}}{(2n+2)!}.
\]

Hence \(\sin(x^2)\) is approximated by

\[p_{2n+1}(x^2) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!}x^{4k+2}
\]

on \([0, \frac{\pi}{2}]\) with error \(\leq \frac{(\frac{\pi}{2})^{4n+4}}{(2n+2)!}\). The limit of these error terms is 0 as \(n \to \infty\) (use Theorem 17.7, page 169 combined with Exercise 17.17(a), page 173, using \(k = (\frac{\pi}{2})^2\), therefore

\[
\sin(x^2) = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!}x^{2k+1} = f(x).
\]
Homework for Section 35, due ??? (only the starred problems will be graded):

4*(a, b, c), 15*(a, b, c), 16*(a, b), 19*(a, c, e, g)

Hints:

Hint for proving non-uniform convergence: Definition 35.6 says that \((f_n)\) converges uniformly to \(f\) on \(S\) if and only if for all \(\epsilon > 0\) there exists \(N > 0\) such that \(n, m > N\) implies \(|f_n(x) - f(x)| < \epsilon\) for all \(x \in S\). Therefore \((f_n)\) fails to converge uniformly to \(f\) on \(S\) if and only if there exists at least one \(\epsilon > 0\) such that for all \(N\) one can find at least one \(n \in \mathbb{N}\) and one \(x \in S\) such that \(n > N\) and \(|f_n(x) - f(x)| \geq \epsilon\). The latter statement is equivalent to saying that there exists at least one \(\epsilon > 0\) such that for all \(N \in \mathbb{N}\) one can find at least one \(x \in S\) such that \(|f_N(x) - f(x)| \geq \epsilon\).

4(b): Use Definition 35.6.

4(c): Use \(\epsilon = \frac{1}{2}\). For each \(N \in \mathbb{N}\) find an appropriate \(x\).

15(b): Let \(M_n\) denote \(\sup_{x \in S} |f_n(x)|\) for each \(n \in \mathbb{N}\). By the Cauchy Convergence Criterion, there exists \(N\) such that \(m, n > N\) implies \(|f_m(x) - f_n(x)| < 1\) for all \(x \in S\). Use this to argue that \(n > N + 1\) implies \(|f_n(x)| < 1 + M_{N+1}\) using a triangle inequality argument. Therefore, for all \(n \in \mathbb{N}\), \(|f_n(x)| \leq \max\{M_1, M_2, \ldots, M_N, 1 + M_{N+1}\}\).

15(c): Consider \(f_n(x) = \frac{1}{x + \frac{1}{n}}\) on \(S = (0, 1]\). Find the formula for \(f(x)\), prove that \((f_n)\) converges pointwise to \(f\), prove that each \(f_n\) is bounded, and prove that \(f\) is unbounded (all with reference to \(S = (0, 1]\)).

16(b): Use \(\epsilon = 1\). For each \(N \in \mathbb{N}\) find an appropriate \(x\).

19(a): Apply the Weierstrass M-test with \(M_n = \frac{1}{n^2}\).

19(c): Apply the Weierstrass M-test with \(M_n = \frac{1}{n\sqrt{2}}\) and use the \(p\)-test or the integral comparison test.

19(e): You cannot apply the Weierstrass M-test to prove non-uniform convergence. Try proving that \((f_n)\) fails the Cauchy Criterion for Function Sequences (Comment 7 above or Theorem 35.10 in the textbook), where

\[ f_n(x) = \sum_{k=1}^{n} \frac{x^2}{k^2}. \]
This amounts to proving that there exits $\epsilon > 0$ such that for all $N$ there exist $n > m > N$ and $x \in [5, \infty)$ such that

$$|f_n(x) - f_m(x)| = \left| \frac{x^2}{(m+1)^2} + \cdots + \frac{x^2}{n^2} \right| \geq \epsilon.$$

It will suffice to prove that there exits $\epsilon > 0$ such that for all $N \in \mathbb{N}$ there exists $x \in [5, \infty)$ such that

$$\left| \frac{x^2}{N^2} \right| \geq \epsilon.$$