1. Let \((a_n)\) be a sequence of numbers. Let \(s_1 = a_1, s_2 = a_1 + a_2, s_3 = a_1 + a_2 + a_3, \ldots, s_n = \sum_{i=1}^{n} a_i\). The sequence \((s_n)\) is called the infinite series corresponding to the sequence \((a_n)\).

2. If \((s_n)\) converges to a real number \(s\) then we write \(\sum_{i=1}^{\infty} a_i = \lim s_n = s\). If \(\lim s_n = +\infty\) then we write \(\sum_{i=1}^{\infty} a_i = +\infty\) and say that the infinite series diverges to infinity. By an abuse of notation (i.e. inconsistent notation), we refer to \(\sum_{i=1}^{\infty} a_i\) as an infinite series.

3. See Examples 32.1 and 32.7 and Practice 32.3 for examples of convergent infinite series and their limits.

4. The harmonic series \(\sum_{n=1}^{\infty} \frac{1}{n}\) diverges to infinity. Reason: note that \(2^k + 2^k = 2 \cdot 2^k = 2^{k+1}\). The sequence of partial sums \(s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}\) satisfies

\[
s_{2^{k+1}} - s_{2^k} = \frac{1}{2^k + 1} + \frac{1}{2^k + 2} + \cdots + \frac{1}{2^{k+1}} \geq \frac{1}{2^{k+1} + 2^k} + \frac{1}{2^{k+1} + 2^k} + \cdots + \frac{1}{2^{k+1} + 2^k} = \frac{2^k}{2^{k+1}} = \frac{1}{2}.
\]

Therefore, there exists \(\epsilon > 0\) such that for all \(N\) there exists \(m, n\) such that \(m, n > N\) and \(|s_m - s_n| \geq \epsilon\): just use \(\epsilon = \frac{1}{2}\) and choose \(m\) and \(n\) so that \(m = 2^{k+1}\) and \(n = 2^k\) and \(2^k > N\). Therefore the sequence \((s_n)\) is not Cauchy and cannot converge. Since \((s_n)\) is monotone increasing and divergent, it cannot be bounded. Therefore \(\sum_{n=1}^{\infty} \frac{1}{n} = \lim s_n = +\infty\).

5. Let \((a_n)\) be a sequence and let \((s_n)\) be its infinite series. Then \(\sum_{i=1}^{\infty} a_i\) exists as a real number if and only if \((s_n)\) is Cauchy if and only if for all \(\epsilon > 0\) there exists \(N\) such that \(n > m > N\) implies

\[|s_m - s_n| = |a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon.
\]

This is called the Cauchy Criterion for Series.

6. If \((a_n)\) is a sequence and \(\sum_{i=1}^{\infty} a_i\) exists as a real number then \(\lim a_n = 0\).

Proof: Let \(\epsilon > 0\) be given. Then there exists \(N\) such that \(n > m > N\) such that \(|a_{m+1} + \cdots + a_n| < \epsilon\). In particular, setting \(n = m + 1\) we have \(n > N + 1\) implies \(|a_n| < \epsilon\).
7. The harmonic series (comment 4 above) shows that it is possible for \( \lim_{n \to \infty} a_n = 0 \) yet \( \sum_{i=1}^{\infty} a_i = +\infty \). So the converse of comment 6 is false.

8. The infinite series \( \sum_{i=n}^{\infty} \frac{1}{n^2} \) exists as a real number. Proof: we just need to show that it is Cauchy. For \( n > m \) we have

\[
|a_{m+1} + \cdots + a_n| = \frac{1}{(m+1)^2} + \cdots + \frac{1}{n^2} = U(f, P)
\]

where \( f : [m+1, n+1] \to \mathbb{R} \) is defined by \( f(x) = \frac{1}{x^2} \) and

\[
P = \{m+1, m+2, \ldots, n+1\}.
\]

For \( n > m > \frac{1}{\epsilon} \) we have

\[
U(f, P) \geq U(f) = \int_{m+1}^{n+1} \frac{1}{x^2} \, dx = -\frac{1}{x}\bigg|_{m+1}^{n+1} = -\frac{1}{n+1} + \frac{1}{m+1} < \frac{1}{m+1} < \frac{1}{m} < \epsilon.
\]

In fact, it can be shown that \( \sum_{i=n}^{\infty} \frac{1}{n^2} \approx 1.644934066848226 \).

Homework for Section 32, due ??? (only the starred problems will be graded):

1, 2, 4\star(a, b, c), 5\star(d, f), 7\star, 9\star, 11\star, 13\star(a, b, c)

Hints:

4(a): Consider the sequence of partial sums.
4(b): Prove by contradiction.
4(c): Show that the sequence \((s_n)\) diverges to \(+\infty\). Compare \((s_n)\) with the sequence of partial sums of the harmonic series.

5(f): Expand \( \frac{1}{(2i-1)(2i+1)} \) into \( \frac{a}{2i-1} + \frac{b}{2i+1} \) using partial fraction decomposition, where \( a \) and \( b \) are appropriate constants, then observe that \( s_n = \sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} \) is a telescoping sum which simplifies nicely. Now compute \( \lim_{n \to \infty} s_n \).

7: Apply the Cauchy Criterion for Series to one of the series, assuming that it holds for the other series.

9: You can use the hint in the back of the book, or you can use logic similar to that in Problem 4(c).
11: Use the Cauchy Criterion for Series combined with $|b_n| \leq M$ for some $M > 0$ and the Triangle Inequality.

13(a): Compute $s_n = \sum_{i=1}^{n} b_n$ using limit properties, assuming that $\sum a_n$ and $\sum b_n$ are convergent.

13(b): Start with a known convergent series and pull it apart into two divergent series.

13(c): Prove the contrapositive: if $\sum c_n$ is convergent, where $c_n = a_{2n-1} + a_{2n}$ and $a_k \geq 0$ for all $k$, then $\sum a_n$ is convergent. Use the monotone convergence theorem. Outline: If $\sum c_n$ is convergent, then the series $\sum a_n$ is monotone increasing and bounded, therefore convergent. (It’s easy to show that the sequence $s_1, s_2, s_3, \ldots$ is monotone increasing, where $s_n = a_1 + a_2 + \cdots + a_n$. It’s also easy to show that the sequence $s_2, s_4, s_6, \ldots$ is bounded by comparing it with the sequence $t_1, t_2, t_3, \ldots$, where $t_n = c_1 + c_2 + \cdots + c_n$. Use this to show that the sequence $s_1, s_3, s_5, \ldots$ is also bounded. Therefore the sequence $s_1, s_2, s_3, \ldots$ is bounded. Work out the details.)