Math 316-01 Intermediate Analysis

Questions for Section 29: The Riemann Integral

1. Some preliminaries: a partition of \([a, b]\) is \(P = \{x_0, x_1, \ldots, x_n\}\), where \(x_0 = a\), \(x_n = b\), and \(x_0 < x_1 < \cdots < x_n\). Given \(i, 1 \leq i \leq n\), we set \(\Delta x_i = x_i - x_{i-1}\) (length of the subinterval \([x_{i-1}, x_i]\)). A refinement of a partition \(P\) is a partition \(Q\) where \(P \subseteq Q\). For example, \([a, b] = [1, 9]\), \(P = \{1, 3, 4, 7, 9\}\), \(\Delta_1 = 2\), \(\Delta_2 = 1\), \(\Delta_3 = 3\), \(\Delta_4 = 2\), \(Q = \{1, 2, 3, 4, 7, 8, 9\}\).

2. Given a function \(f : [a, b] \to \mathbb{R}\) such that \(f([a, b])\) is bounded, and given a partition \(P = \{x_0, x_1, \ldots, x_n\}\) of \([a, b]\), we set \(M_i = \text{sup}(f([x_{i-1}, x_i]))\) and \(m_i = \text{inf}(f([x_{i-1}, x_i]))\). For example, if \(f : [1, 9] \to \mathbb{R}\) is given by \(f(x) = x^2\), and if \(P = \{1, 3, 4, 7, 9\}\), then \(M_1 = 9\), \(M_2 = 16\), \(M_3 = 49\), \(M_4 = 81\), \(m_1 = 1\), \(m_2 = 9\), \(m_3 = 16\), \(m_4 = 49\).

3. Given a bounded function \(f : [a, b] \to \mathbb{R}\) and a partition \(P = \{x_0, x_1, \ldots, x_n\}\) of \([a, b]\), we set \(U(f, P) = M_1 \Delta_1 + \cdots + M_n \Delta_n\) (the upper sum) and \(L(f, P) = m_1 \Delta_1 + \cdots + m_n \Delta_n\) (the lower sum). We always have \(L(f, P) \leq U(f, P)\). In our example above, \(U(f, P) = 9 \cdot 2 + 16 \cdot 1 + 49 \cdot 3 + 81 \cdot 2 = 343\) and \(L(f, P) = 1 \cdot 2 + 9 \cdot 1 + 16 \cdot 3 + 49 \cdot 2 = 157\).

4. If \(f : [a, b] \to \mathbb{R}\) is a bounded function and \(P\) and \(Q\) are partitions of \([a, b]\) such that \(P \subseteq Q\), then \(U(f, P) \geq U(f, Q)\).

Proof: \(Q\) is obtained by adding partition points to \(P\). We will prove the result assuming that \(Q\) contains one more point than \(P\). So consider \(Q = P \cup \{y\}\) where \(x_{i-1} < y < x_i\). The only difference between \(U(f, P)\) and \(U(f, Q)\) is that the term \(\text{sup}(f([x_{i-1}, x_i]))(x_i - x_{i-1})\) in \(U(f, P)\) is replaced by \(\text{sup}(f([x_{i-1}, y]))(y - x_{i-1}) + \text{sup}(f([y, x_i]))(x_i - y)\) in \(U(f, Q)\). However, \(\text{sup}(f([x_{i-1}, x_i])) \geq \text{sup}(f([x_{i-1}, y]))\) and \(\text{sup}(f([x_{i-1}, x_i])) \geq \text{sup}(f([y, x_i]))\), therefore

\[
\text{sup}(f([x_{i-1}, x_i]))(x_i - x_{i-1}) = \text{sup}(f([x_{i-1}, x_i]))(y - x_{i-1}) + \text{sup}(f([x_{i-1}, x_i]))(x_i - y)
\]

\[
\geq \text{sup}(f([x_{i-1}, y]))(y - x_{i-1}) + \text{sup}(f([y, x_i]))(x_i - y).
\]

Since one term in \(U(f, P)\) is replaced by a smaller sum of two terms in \(U(f, Q)\), we must have \(U(f, P) \geq U(f, Q)\).

5. If \(f : [a, b] \to \mathbb{R}\) is a bounded function and \(P\) and \(Q\) are partitions of \([a, b]\) such that \(P \subseteq Q\), then \(L(f, P) \geq L(f, Q)\).
**Proof:** The proof is similar to that above. The infimum of \( f \) over \([x_{i-1}, x_i]\) is \( \leq \) the infimum of \( f \) over \([x_{i-1}, y]\) and over \([y, x_i]\).

6. If \( f : [a, b] \to \mathbb{R} \) is a bounded function and \( P \) and \( Q \) are arbitrary partitions of \([a, b]\), then \( L(f, P) \leq U(f, Q) \).

**Proof:** Note that \( P \cup Q \) is a refinement of \( P \) and a refinement of \( Q \). So we have \( L(f, P) \leq L(f, P \cup Q) \leq U(f, P \cup Q) \leq U(f, Q) \), combining the results in comments 4, and 5 above.

7. Let \( f : [a, b] \to \mathbb{R} \) be a bounded function. We can see that the set \( \{U(f, P) : P \vdash [a, b]\} \) is bounded below by every \( L(f, Q) \). Set \( U(f) = \inf \{U(f, P) : P \vdash [a, b]\} \). We can also see that the set \( \{L(f, Q) : Q \vdash [a, b]\} \) is bounded above by every \( U(f, P) \). Set \( L(f) = \sup \{L(f, Q) : Q \vdash [a, b]\} \). Then we have \( L(f, P) \leq U(f) \) for all \( P \), therefore \( L(f) \leq U(f) \). When \( L(f) < U(f) \) then we say that \( f \) is not integrable over \([a, b]\). But when \( L(f) = U(f) \) then we say that \( f \) is integrable over \([a, b]\), and we define

\[
\int_a^b f = L(f) = U(f).
\]

8. An example of a non-integrable function \( g : [0, 2] \to \mathbb{R} \) is given in Example 29.8, page 273. We have \( L(g) = 0, U(g) = 2 \).

9. We will comment on Example 29.7, page 272. Let \( f : [0, 1] \to \mathbb{R} \) be defined by \( f(x) = x^2 \). Then \( L(f) = U(f) = \frac{1}{3} \). To see this, let \( P_n \) denote the partition \( \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\} \). Then

\[
L(f, P_n) = \frac{1}{3} \left( \frac{n - 1}{n} \right) \left( \frac{2n - 1}{2n} \right).
\]

and

\[
U(f, P_n) = \frac{1}{3} \left( \frac{n + 1}{n} \right) \left( \frac{2n + 1}{2n} \right).
\]

We

\[
L(f) = \sup \{L(f, P) : P \vdash [0, 1]\} \geq \sup \{L(f, P_n) : n \in \mathbb{N}\} = \frac{1}{3}
\]

and

\[
U(f) = \inf \{U(f, P) : P \vdash [0, 1]\} \leq \inf \{U(f, P_n) : n \in \mathbb{N}\} = \frac{1}{3},
\]
therefore \( L(f) \geq U(f) \). But we always have \( L(f) \leq U(f) \), therefore \( L(f) = U(f) \). Moreover \( \frac{1}{3} \leq L(f) \leq \frac{1}{3} \), which can only occur of \( L(f) = U(f) = \frac{1}{3} \). Therefore \( \int_0^1 f = \frac{1}{3} \).

10. We will comment on the proof of Theorem 29.9. This theorem is to be interpreted as an alternative definition of integrability, which should be useful for proofs in later sections. Assume \( f : [a, b] \to \mathbb{R} \) is bounded and integrable on \([a, b]\). Then \( L(f) = U(f) = \int_a^b f \). Let \( \epsilon > 0 \) be given. Then there must exist a partition \( P \) such that \( U(f, P) - L(f, P) < \epsilon \). To see this choose a partition \( P_1 \) such that \( L(f) - \frac{\epsilon}{2} < L(f, P_1) \leq L(f) \) and choose a partition \( P_2 \) such that \( U(f) \leq U(f, P_2) < U(f) + \frac{\epsilon}{2} \). Then we have

\[
L(f) - \frac{\epsilon}{2} < L(f, P_1 \cup P_2) \leq U(f, P_1 \cup P_2) < U(f) + \frac{\epsilon}{2},
\]

which implies that \( L(f, P_1 \cup P_2) \) and \( U(f, P_1 \cup P_2) \) are trapped between \( \int_a^b f - \frac{\epsilon}{2} \) and \( \int_a^b f + \frac{\epsilon}{2} \). This means that the gap between \( L(f, P_1 \cup P_2) \) and \( U(f, P_1 \cup P_2) \) is smaller than \( \epsilon \). Hence integrability implies we can find a partition \( P \) such that \( U(f, P) - L(f, P) < \epsilon \).

Conversely, suppose that \( f : [a, b] \to \mathbb{R} \) is a bounded function that meets this criterion. We will prove that \( f \) is integrable over \([a, b]\). For all \( n \in \mathbb{N} \) there exists \( P_n \) such that \( 0 \leq U(f, P_n) - L(f, P_n) < \frac{1}{n} \). This implies \( 0 \leq U(f) - L(f, P_n) < \frac{1}{n} \). Hence \( \lim L(f, P_n) = U(f) \). We also have \( 0 \leq U(f, P_n) - L(f) < \frac{1}{n} \). This implies \( \lim U(f, P_n) = L(f) \). We also have \( \lim (U(f, P_n) - L(f, P_n)) = 0 \). Using limit properties, this implies \( U(f) - L(f) = 0 \). Therefore \( L(f) = U(f) \) and \( f \) is integrable.

Homework for Section 29, due ???: (only the starred problems will be graded):

1, 2, 7*, 8*, 9*, 13*, 16*, 29*

Hints:

7. Mimic Example 29.7, page 272 and Comment 9 of these notes. Use \( 1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}n^2(n + 1)^2 \).

8. Let \( f : [0, 1] \to \mathbb{R} \) be defined by

\[
f(x) = \begin{cases} 
1 & x \in \mathbb{Q} \\
-1 & x \notin \mathbb{Q} \\
\end{cases}
\]
Show that \( L(f) < U(f) \) as in Example 29.8, page 273. Then show that \( f^2 : [0, 1] \to \mathbb{R} \) defined by \( f^2(x) = f(x)^2 = 1 \) is integrable with \( L(f) = U(f) = 1 \).

9. You should be able to construct a counterexample using \( h : [0, 1] \to \mathbb{Q} \) defined by
\[
h(x) = \begin{cases} 
  1 & x \in \mathbb{Q}, \\
  -1 & x \not\in \mathbb{Q}.
\end{cases}
\]
Compare with Example 29.8, page 273.

13. Prove one case of the contrapositive, namely that if \( f(c) > 0 \) for some \( c \in [a, b] \) then \( L(f) > 0 \). Note that by continuity of \( f \) at \( c \) there exists a \( \delta > 0 \) such that \( x \in [a, b] \) and \( c - \delta < x < c + \delta \) implies \(|f(x) - f(c)| < |f(c)|\), which implies \( f(x) > 0 \) for these values of \( x \). Now construct a partition \( P \) which takes advantage of this fact, so that \( L(f, P) > 0 \). Be specific about the contents of \( P \). This implies \( L(f) \geq L(f, P) > 0 \). It will help to draw a diagram first.

16. To make the problem more concrete and manageable, assume that \( f : [0, 10] \to \mathbb{R} \) is defined by
\[
f(x) = \begin{cases} 
  100 & x = 2 \\
  200 & x = 5 \\
  0 & x \in [0, 10] \setminus \{2, 5\}.
\end{cases}
\]
It should be clear that \( L(f) \geq 0 \). So it will suffice to show that \( U(f) = 0 \). This will imply that \( U(f) \leq L(f) \), hence \( L(f) = U(f) = 0 \), which implies that \( f \) is integrable over \([0, 10]\) and \( \int_0^{10} f = 0 \). To show that \( U(f) = 0 \), show that for all \( \epsilon > 0 \) there exists a partition \( P \) such that \( U(f, P) < \epsilon \). Then \( U(f) = \inf \{U(f, P) : P \vdash [0, 10]\} = 0 \). You should construct the partition in such a way that \( 2 \in [x_1, x_2], 5 \in [x_3, x_4], \) and the size of these intervals is small enough to force \( U(f, P) < \epsilon \). It will help to draw a diagram first.