Math 316-01 Intermediate Analysis

Questions for Section 27: L’Hospital’s Rule

1. Assume \( \lim_{x \to c} f(x) = 0 \) and \( \lim_{x \to c} g(x) = 0 \). If the limit \( \lim_{x \to c} \frac{f(x)}{g(x)} \) exists as a finite number, then it is said to have indeterminate form \( \frac{0}{0} \). The actually limit can be any real number \( k \), as we can see by setting \( f(x) = k(x - c) \) and \( g(x) = x - c \).

2. Cauchy Mean Value Theorem (27.1): Let \( f : [a, b] \to \mathbb{R} \) and \( g : [a, b] \to \mathbb{R} \) both satisfy the hypotheses of the Mean Value Theorem (continuous on \([a, b]\), differentiable on \((a, b)\)). Then there exists \( c \in (a, b) \) such that \( (f(b) - f(a))g'(c) = (g(b) - g(a))f'(c) \). \textbf{Proof:} This is equivalent to showing that \( (f(b) - f(a))g'(c) - (g(b) - g(a))f'(c) = 0 \). If we set \( h(x) = (f(b) - f(a))g(x) - (g(b) - g(a))f(x) \) then we want to prove that \( h'(c) = 0 \) for some \( c \in (a, b) \). Since \( h : [a, b] \to \mathbb{R} \) is continuous on \([a, b]\), differentiable on \((a, b)\), and satisfies \( h(a) = h(b) \), Rolle’s Theorem (26.2, page 242) says that \( c \) exists.

3. L’Hospital’s Rule (27.2): Let \( f : [a, b] \to \mathbb{R} \) and \( g : [a, b] \to \mathbb{R} \) both satisfy the hypotheses of the Mean Value Theorem (continuous on \([a, b]\), differentiable on \((a, b)\)). Assume \( c \in [a, b] \) and \( f(c) = g(c) = 0 \). Assume that \( g'(x) \neq 0 \) for all \( x \in (a, b) \cap N^*(c, \delta) \), i.e. at all \( x \) in the vicinity of \( c \) in \((a, b)\) except possibly at \( c \) itself. If \( \lim_{x \to c} \frac{f'(x)}{g'(x)} \) exists as a real number, then \( \lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)} \). \textbf{Proof:} Write \( \lim_{x \to c} \frac{f(x)}{g(x)} = L \). Let \( (x_n) \subseteq (a, b) \cap N^*(c, \delta) \) satisfy \( \lim x_n = c \). We will show \( \lim \frac{f(x_n)}{g(x_n)} = L \). Since \( f \) and \( g \) are continuous on \([x_n, c]\) and differentiable on \((x_n, c)\), the Cauchy Mean Value Theorem says there exists \( c_n \in (x_n, c) \) such that \( (f(c) - f(x_n))g'(c_n) = (g(c) - g(x_n))f'(c_n) \). That is, \( f(x_n)g'(c_n) = g(x_n)f'(c_n) \). We wish to rearrange this expression algebraically by dividing by both \( g'(c_n) \) and \( g(x_n) \). By hypothesis, \( g'(c_n) \neq 0 \). We must show that \( g(x_n) \neq 0 \). If in fact \( g(x_n) = g(c) = 0 \), then by Rolle’s Theorem we must have \( g'(d) = 0 \) for some \( d \in (x_n, c) \), which contradicts the fact that \( g'(x) \neq 0 \) in the vicinity of \( c \) (except possibly at \( c \) itself). So \( g(x_n) \neq 0 \). Hence we have \( \frac{f(x_n)}{g(x_n)} = \frac{f'(c_n)}{g'(c_n)} \). Since \( x_n \to c \) and \( x_n < c_n < c \) for all \( n \), we have \( c_n \to c \). We know that \( \frac{f'(c_n)}{g'(c_n)} \) approaches \( L \) as \( n \to \infty \), therefore \( \frac{f(x_n)}{g(x_n)} \) approaches \( L \) as \( n \to \infty \).

4. See Examples 27.3 and 27.4 (page 253) for applications of L’Hospital’s Rule. In Example 27.4, the rule is applied twice to compute the limit.
5. **L’Hospital’s Rule (27.8):** Let \( f : (b, \infty) \to \mathbb{R} \) and \( g : (b, \infty) \to \mathbb{R} \) both be differentiable on \((b, \infty)\). Suppose that \( \lim_{x \to \infty} f(x) = \infty \) and \( \lim_{x \to \infty} g(x) = \infty \) and \( g'(x) \neq 0 \) for all \( x \in (b, \infty) \). If \( \lim_{x \to \infty} \frac{f'(x)}{g'(x)} \) exists as a real number, then \( \lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)} \). **Proof:** Assume \( \lim_{x \to \infty} \frac{f'(x)}{g'(x)} = L \). We will prove that \( \lim_{x \to \infty} \frac{f(x)}{g(x)} = L \). The hypotheses guarantee that we can find a real number \( x_0 > b \) such that for \( x > x_0 \) we have \( f(x) > 0 \) and \( g(x) > 0 \). This is useful if we want to divide by these terms. The hypotheses also guarantee that we can find a real number \( x_1 > x_0 \) such that for all \( x > x_1 \) we have \( f(x) > f(x_0) \) and \( g(x) > g(x_0) \). This is useful if we wish to divide by \( f(x) - f(x_0) \) and by \( g(x) - g(x_0) \). For any \( x > x_1 \), the Cauchy Mean Value Theorem guarantees that we can find a \( c_x \in (x_0, x) \) such that \( (f(x) - f(x_0))g'(c_x) = (g(x) - g(x_0))f'(c_x) \), which is equivalent to \( \frac{f'(c_x)}{g'(c_x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} \). Solving for \( \frac{f(x)}{g(x)} \), we have \( f(x) = g(x) \frac{f(x)}{g(x)} \left( \frac{1}{1 - \frac{g(x_0)}{g(x)}} \right) \). To force \( \left| \frac{f(x)}{g(x)} - L \right| < \epsilon \), \( x_0 \) is chosen large enough to make \( c_x \) large enough so that \( \frac{f(x)}{g(x)} \) is sufficiently close to \( L \), and \( x_1 \) is chosen sufficiently large so that \( 1 - \frac{g(x_0)}{g(x)} \) is sufficiently close 1. A more formal argument using the triangle inequality and \( \frac{\epsilon}{2} \) is given in the proof of Theorem 27.8, page 255.

6. See Practice 27.9, Example 27.10, and Example 27.11 (page 256) for examples of the use of this second form of L’Hospital’s Rule. The trick to using either form of this rule is to convert all indeterminate forms to either \( \frac{0}{\infty} \) or \( \frac{\infty}{\infty} \), then carefully verifying that the hypotheses of L’Hospital’s Rule are met before applying it. Sometimes this requires using the logarithm function as in Example 27.11 combined with continuity of the function \( f(x) = e^{x^2} \) (not proved until later in the course).

**Homework for Section 27, due ???** (only the starred problems will be graded):

1, 2, 3\(^*(f, h, j)\), 4\(^*(b, g, h)\), 5\(^*\), 9\(^*(a, b)\), 10\(^*\), 11\(^*(a)\)

**Hints:**

3(f): \((1 + x)^{\frac{1}{2}} = e^{\ln(1 + x)/2} = e^{\frac{1}{2} \ln 1 + x}\). Use L’Hospital’s rule on \( \frac{1}{x} \ln 1 + x \), then use continuity of \( f(x) = e^x \) as in Example 27.11.

3(h): \( \frac{1}{x} - \frac{1}{\sin x} = \frac{\sin x - x}{x \sin x} \). Use L’Hospital’s rule on the latter expression.

3(j): Use L’Hospital’s rule \( n \) times.
4(g): Similar to 3(f).

4(h): \( x^{2x} = e^{\ln x^{2x}} = e^{2x \ln x} = e^{\ln x} \). Similar to 3(f).

9(a,b): \( \lim_{x \to \infty} f(x) = L \) if and only if for all \( \epsilon > 0 \) there exists \( N \) such that \( x > N \) implies \( |f(x) - L| < \infty \). Compare this to Definition 16.2, page 158. Now adapt the proof of Theorem 17.1(a,c), page 166, accordingly.

10. \( \lim_{x \to \infty} f(x) = \infty \) if and only if for all \( M \) there exists \( N \) such that \( x > N \) implies \( f(x) > M \). Assuming this as a given, let \( \epsilon > 0 \) be given. You must show there exists \( N \) such that \( x > N \) implies \( \left| \frac{k}{f(x)} \right| < \epsilon \).

11(a). Construct \( f \) and \( g \) using \( 1 + \frac{1}{2} \sin(\frac{1}{x}) \) as a building-block.