1. Let \( f : D \to \mathbb{R} \) be a function. Define formally what it means for \( f \) to be continuous at \( c \in D \).

2. Let \( f : D \to \mathbb{R} \) be a function. Define formally what it means for \( f \) to be uniformly continuous on \( D \).

3. If \( f : D \to \mathbb{R} \) is uniformly continuous on \( D \), why is it continuous at all \( c \in D \)?

4. Give the negation of the definition of uniform continuity in Question 2.

5. I will prove that \( f : (0, \infty) \to \mathbb{R} \) defined by \( f(x) = \frac{1}{x} \) is not uniformly continuous. We have to produce an \( \epsilon > 0 \) for which the definition fails. I will choose (arbitrarily) \( \epsilon = .1 \). Consider any \( \delta > 0 \). Let \( y = x + \frac{\delta}{2} \). Then \( |x - y| = \frac{\delta}{2} < \delta \). I will produce \( x \) so that \( |f(x) - f(y)| > \epsilon \). We have 
   \[
   |f(x) - f(y)| = \left| \frac{1}{x} - \frac{1}{x + \frac{\delta}{2}} \right| = \frac{\delta}{x(x + \frac{\delta}{2})} \geq \frac{\delta}{2x},
   \]
   which will be \( \geq \epsilon \) when \( x \leq \sqrt{\frac{5}{2}} \).

So the definition of uniformly continuous fails for \( \epsilon = .1 \), \( x = \sqrt{\frac{5}{2}} = \sqrt{5\delta}, y = \sqrt{5\delta} + \frac{\delta}{2} \). Therefore \( f \) is not uniformly continuous on \( (0, \infty) \).

6. I will prove that \( f : [5, \infty) \to \mathbb{R} \) defined by \( f(x) = \frac{1}{x} \) is uniformly continuous. Let \( \epsilon > 0 \) be given. We must produce \( \delta > 0 \) such that \( |x - y| < \delta \) guarantees \( |f(x) - f(y)| < \epsilon \). Note that \( x, y \in [5, \infty) \) guarantees \( xy \geq 25 \).

We will use \( \delta = 25\epsilon \). Then \( |x - y| < 25\epsilon \) guarantees \( |f(x) - f(y)| = \left| \frac{1}{x} - \frac{1}{y} \right| = \frac{|y - x|}{xy} = |y - x| \cdot \frac{1}{xy} \leq 25\epsilon \cdot \frac{1}{25} = \epsilon \). Therefore \( f \) is uniformly continuous on \( [5, \infty) \). A similar argument shows that \( f(x) = \frac{1}{x} \) is uniformly continuous on \( [a, \infty) \) for each \( a > 0 \) (using \( \delta = a^2 \epsilon \)).

7. Theorem 23.6 shows that continuity implies uniform continuity on a compact set. In other words, if \( D \) is compact and \( f : D \to \mathbb{R} \) is continuous, then \( f \) is uniformly continuous. Proof: Let \( \epsilon > 0 \) be given. We will cover \( D \) with open sets as follows: Let \( c \in D \) be given. By continuity, we know that there exists \( \delta_c > 0 \) (depending on \( c \)) such that \( x \in D \) and \( |x - c| < \delta_c \) implies \( |f(x) - f(c)| < \frac{\epsilon}{2} \). In other words, \( x \in N(c, \delta_c) \cap D \Rightarrow |f(x) - f(c)| < \frac{\epsilon}{2} \).

The sets in \( \{ N(c, \frac{\delta_c}{2}) : c \in D \} \) form an open cover of \( D \). By compactness, we can say \( D \subseteq N(c_1, \frac{\delta_1}{2}) \cup N(c_2, \frac{\delta_2}{2}) \cup \cdots \cup N(c_k, \frac{\delta_k}{2}) \) for some finite collection \( c_1, c_2, \ldots, c_k \in \bar{D} \). Let \( \delta = \min\{\delta_{c_1}, \ldots, \delta_{c_k}\} \). If \( |x - y| < \frac{\delta}{2} \), say
that \( x \in N(c_i, \frac{\delta_i}{2}) \) for some \( i \). Then \( |x - c_i| \leq \frac{\delta_i}{2} \). Therefore \( |y - c_i| = |y - x + x - c_i| \leq |y - x| + |x - c_i| \leq \frac{\delta}{2} + \frac{\delta}{2} \leq \delta_i \). So both \( x \) and \( y \) belong to \( N(c_i, \delta_i) \). This implies \( |f(x) - f(y)| = |f(x) - f(c_i) + f(c_i) - f(y)| \leq |f(x) - f(c_i)| + |f(y) - f(c_i)| \leq \frac{\delta}{2} + \frac{\delta}{2} = \epsilon \).

8. Exercise 23.14 supplies an alternative proof of Theorem 23.6. Suppose \( f \) is not uniformly continuous on \( D \). Then there exists \( \epsilon > 0 \) for which the definition of uniform convergence fails. Hence for every \( n \) we can find \( x_n, y_n \in D \) such that \( |x_n - y_n| < \frac{1}{n} \) yet \( |f(x_n) - f(y_n)| \geq \epsilon \). Since \( D \) is compact, the sequence \( (x_n) \) has a convergent subsequence \( (x_{n_k}) \) with a limit \( d \) in \( D \). Write \( x_{n_k} \to d \). Then \( f(x_{n_k}) \to f(d) \) by continuity of \( f \) on \( D \). Note that we have \( |y_{n_k} - d| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - d| \), and both quantities on the right hand side of the inequality can be made arbitrarily small by choosing \( k \) sufficiently large, hence \( y_{n_k} \to d \) also. This implies that \( f(y_{n_k}) \to f(d) \) by continuity of \( f \) on \( D \). Yet \( |f(y_{n_k}) - f(d)| = |(f(y_{n_k}) - f(x_{n_k})) - (f(d) - f(x_{n_k}))| \geq |(f(y_{n_k}) - f(x_{n_k})) - f(d) - f(x_{n_k})| \geq \epsilon - |f(d) - f(x_{n_k})| \), and for \( k \) sufficiently large the last expression is \( \geq \frac{\epsilon}{2} \). This contradicts \( f(y_{n_k}) \to f(d) \). Therefore \( f \) must be uniformly continuous on \( D \).

9. Theorem 23.8 says that if \( f : D \to \mathbb{R} \) is uniformly continuous and \( (x_n) \subseteq D \) is Cauchy, then \( (f(x_n)) \) is Cauchy. The argument is this: Let \( \epsilon > 0 \) be given. We must find \( N \) such that \( m, n > N \) implies \( |f(x_n) - f(x_m)| < \epsilon \). This looks like something a uniformly continuous function can guarantee: let \( \delta > 0 \) be chosen so that \( |f(x) - f(y)| < \epsilon \) whenever \( |x - y| < \delta \). Since \( (x_n) \) is Cauchy, we can find \( N \) such that \( m, n > N \) implies \( |x_n - x_m| < \delta \). Combining our choice of \( \delta \) with our choice of \( N \), we have \( m, n > N \) implies \( |x_n - x_m| < \delta \) implies \( |f(x_n) - f(x_m)| < \epsilon \).

10. What is wrong with the following alternative proof of Theorem 23.8?

"Assume \( f : D \to \mathbb{R} \) is uniformly continuous, and assume \( (x_n) \subseteq D \) is Cauchy. Since every Cauchy sequence converges, \( (x_n) \) must converge to some \( x \), therefore by continuity \( (f(x_n)) \) converges to \( f(x) \), therefore \( (f(x_n)) \) must be Cauchy because every convergent sequence is Cauchy."

11. The extension of a uniformly continuous function \( f : (a, b) \to \mathbb{R} \) to a continuous (therefore uniformly continuous) function \( \tilde{f} : [a, b] \to \mathbb{R} \) is given in Theorem 23.9. The argument is this: We need only define \( \tilde{f}(a) = p \) and \( \tilde{f}(b) = q \) carefully so that \( \tilde{f} \) is continuous. We need \( \tilde{f}(s_n) \to p \) whenever \( (s_n) \subseteq [a, b] \) converges to \( a \) with \( s_n \neq a \) for each \( n \). We can assume \( (s_n) \) does not take on the value \( b \). Since \( (s_n) \) is convergent, it is...
Cauchy, therefore \((f(s_n))\) is Cauchy by Theorem 23.8, therefore a limit \(p\) exists. There is a problem with ambiguity here: suppose \((s'_n) \subseteq (a, b)\) is another sequence which converges to \(a\). Then \((s'_n)\) is Cauchy, therefore \((f(s'_n))\) is Cauchy, therefore \((f(s'_n))\) converges to some limit \(p'\). Do we set \(\tilde{f}(a) = p\) or \(\tilde{f}(a) = p'\)? How do we know that \(p = p'\)? Answer: The sequence \(s_1, s'_1, s_2, s'_2, s_3, s'_3\ldots\) converges to \(a\), hence it is Cauchy, hence the sequence \(f(s_1), f(s'_1), f(s_2), f(s'_2), f(s_3), f(s'_3)\ldots\) is Cauchy, hence it converges to some limit \(p''\). We know that every subsequence converges to the same limit \(p''\). One subsequence converges to \(p\), and another subsequence converges to \(p'\), therefore \(p = p'' = p'\). (This subtlety seems to be overlooked in the textbook.) The definition of \(\tilde{f}(b) = q\) follows the same format.

**Homework for Section 23, due ??? (only the starred problems will be graded):**

1, 2, 3*(c, d, f, g), 4*(c), 6*, 10*, 11*

**Hints:**

3(d). Use the information in 3(c).
3(f). Reason as in Question 5 above.
3(g). In Example 21.13 it is shown that \(f : \mathbb{R} \to \mathbb{R}\) defined by

\[
f(x) = \begin{cases} 
  x \sin(\frac{1}{x}) & x \neq 0 \\
  0 & x = 0
\end{cases}
\]

is continuous at all \(x \in \mathbb{R}\). How can you use this information?

4(c). Reason as in Question 6 above.

6. Requires the triangle inequality and \(\frac{\varepsilon}{2}\) somehow.

11. Suppose \(f(D)\) is not bounded. Then one can find \(x_1 \in D\) such that \(f(x_1) > 1\), \(x_2 \in D\) such that \(f(x_2) > 2\), and so on. Explain why the sequence \((x_n)\) has a convergent subsequence \((x_{n_k})\). Explain why \((x_{n_k})\) must be Cauchy. Explain why \(f(x_{n_k})\) must be Cauchy. Explain why \(f(x_{n_k})\) must therefore be bounded. Explain why this contradicts \(f(x_{n_k}) > n_k\) for all \(k \in \mathbb{N}\).