Math 316-01 Intermediate Analysis

Questions for Section 22: Properties of Continuous Functions

1. Let \( f : D \to \mathbb{R} \) be continuous at each \( c \in D \). If \( D \) is compact then \( f(D) = \{ f(c) : c \in C \} \) is compact. Proof: Let \( \mathcal{F} = \{ G_i : i \in I \} \) be an open cover of \( f(D) \). Then \( \{ f^{-1}(G_i) : i \in I \} \) covers \( D \). By Theorem 21.14, we know that \( f^{-1}(G_i) = H_i \cap D \) for some open set \( H_i \). Therefore \( \{ H_i \cap D : i \in I \} \) covers \( D \). Therefore \( \{ H_i : i \in I \} \) is an open cover of \( D \). By compactness of \( D, D \subseteq H_{i_1} \cup \cdots \cup H_{i_n} \). This implies \( D = (H_{i_1} \cap D) \cup \cdots \cup (H_{i_n} \cap D) \). Therefore \( f(c) \in f(D) \) implies \( f(c) \in f(H_{i_k} \cap D) \subseteq G_{i_k} \) for some \( k \leq n \). Thus \( f(D) \) has the finite subcover \( G_{i_1}, \ldots, G_{i_n} \).

2. Let \( f : D \to \mathbb{R} \) be continuous at each \( c \in D \). If \( D \) is compact then \( f(D) \) is compact, therefore \( f(D) \) is closed and bounded. Therefore \( m = \sup f(D) \) exists (why?). Therefore \( m \) is an accumulation point of \( f(D) \) (why?). Therefore \( m \in f(D) \) (why?). Therefore there exists \( c \in D \) such that \( f(c') \leq f(c) \) for all \( c' \in D \) (why?). Many calculus problems are stated in the form “find the maximum value of \( f(x) \).” The existence of the maximum value is guaranteed by continuity of \( f \) and compactness of the domain of \( f \).

3. Give an example of a function \( f : D \to \mathbb{R} \) such that \( f \) is continuous at every \( c \in D \) but there is no \( c \in D \) which satisfies \( f(c') \leq f(c) \) for all \( c' \in C \). Use a bounded set for \( D \) and give a formula for \( f \).

4. Let \( f : [a, b] \to \mathbb{R} \) be continuous at every \( c \in [a, b] \). Then \( f([a, b]) \) is an interval. Proof: by compactness of \([a, b]\), we know that \( f([a, b]) \) is closed and bounded. Therefore, by the logic in question 2, setting \( m = \inf f([a, b]) \) and \( n = \sup f([a, b]) \), we have \( m \in f([a, b]) \) and \( n \in f([a, b]) \), and there exist \( c_1, c_2 \in [a, b] \) such that \( f(c_1) \leq f(c) \leq f(c_2) \) for all \( c \in [a, b] \). We will show that \( f([a, b]) = [f(c_1), f(c_2)] \). Let \( k \in [f(c_1), f(c_2)] \) be given. We must show that \( f(c) = k \) for some \( c \in [a, b] \). We will produce \( c \) using the completeness axiom. Let \( S = \{ x \in [a, b] : f(x) \leq k \} \). Then \( S \) is a non-empty bounded set (why?), therefore it has a least upper bound \( c \) which belongs to \([a, b]\). Suppose \( f(c) \neq k \). Then \( f(c) < k \) or \( f(c) > k \). We will obtain a contradiction. First, we will show that \( f(c) > k \) is ruled out: find a sequence \( (x_n) \) in \( S \) which converges to \( c \) (why can this be done?). By continuity, \( (f(x_n)) \) converges to \( f(c) \). Since each \( f(x_n) \leq k \), we must have \( f(c) \leq k \). (Which theorem in Section 17 justifies this statement?) So apparently \( f(c) < k \). Choose \( \epsilon > 0 \) sufficiently small that \( f(c) + \epsilon < k \). By continuity of \( f \), we can find \( \delta > 0 \)
so that $c - \delta < x < c + \delta$ guarantees $f(c) - \epsilon < f(x) < f(c) + \epsilon$. Therefore $f(c + \tfrac{\delta}{2}) < f(c) + \epsilon < k$. This forces $c + \tfrac{\delta}{2} \in S$, which contradicts the fact that $c$ is an upper bound of $S$. Therefore we must have $f(c) = k$.

5. How can we use the information in Question 4 to prove that there exists a real number $c$ such that $c^5 + c = \sqrt{2}$?

6. Give an example of a function $f : [a, b] \to \mathbb{R}$ such that $f([a, b])$ is not an interval.

Homework for Section 22, due ???: (only the starred problems will be graded):

1, 2, 3\textsuperscript{*}(e), 4\textsuperscript{*}, 7\textsuperscript{*}, 9\textsuperscript{*}, 11\textsuperscript{*}, 13\textsuperscript{*}(a)

Hints:

4. Let $f(x) = 2^x - 3x$. You can assume that this is a continuous function at all $c \in \mathbb{R}$. Use the Intermediate Value Theorem (proved in Question 4).

7. The hint in the back of the book is a good one.

9. What kind of an interval can $f([a, b])$ be, given that it is a subset of $\mathbb{Q}$?

11(a). Prove that $f(x) = |x - p|$ is continuous by cases. First show that it is continuous at every $c > p$, then that it is continuous at $c = p$, then that it is continuous at every $c < p$. Give an $\epsilon-\delta$ proof for each case. For example if $c > p$ then choosing a small enough $\delta$ you can assume that $|x - c| < \delta$ implies that $x > p$, so $f(x) = x - p$. This allows you to drop the absolute value symbols and makes it easier to find $\delta$, given $\epsilon$.

11(b). Why is $f(S)$ compact? Why does this guarantee that $q$ exists?

13(a). Argue by contradiction. If $f$ is neither strictly increasing nor strictly decreasing, there are many possible cases: we could have $x_1 < x_2 < x_3$ with $f(x_1) < f(x_2) < f(x_3) < f(x_1)$ (neither strict increase nor strict decrease) and $f(x_3) < f(x_1)$. Let $k = f(x_3)$. Since $f([x_1, x_2])$ is an interval, it contains all the values between $f(x_1)$ and $f(x_2)$ (it could contain other values as well). Since $k$ belongs to this interval, $k \in f([x_1, x_2])$. Therefore $k = f(x)$ for some $x_1 \leq x \leq x_2$. Since $k = f(x_3)$ and $f$ is injective, $x = x_3$. But this contradicts $x_3 > x_2$. Work out all the other cases.