Math 316-01 Intermediate Analysis

Questions for Section 19: Subsequences

1. Define the term subsequence of \((s_n)\). How are subsequences formed?

2. Let \((s_n)\) be a convergent sequence. Then \((s_n)\) is Cauchy if and only if, for each \(\epsilon > 0\), only finitely many of the differences \(|s_m - s_n| \geq \epsilon\). Given this, is it clear that any subsequence of a Cauchy sequence is Cauchy? Therefore, if a sequence converges, it is Cauchy, hence all subsequences are Cauchy, hence all subsequences converge.

3. In Example 19.5, page 182, it is proved that the sequence \(s_n = x^{\frac{1}{n}}\) converges to 1 when \(0 < x < 1\). The argument is this: the sequence is bounded between 0 and 1, and the terms of the sequence are increasing, therefore there must be a limit \(s\). The same is true of the sequence \(t_n = (x^{\frac{1}{n}})\), having limit \(t\). \((t_n)\) is a subsequence of \(s_n\), so \(t = s\). On the other hand, \(t = \sqrt{s}\), so \(t = s = 1\). Question: in what way is the sequence \((t_n)\) a subsequence of \((s_n)\)? In other words, which terms from \((s_n)\) were dropped to form \((t_n)\)?

4. In Theorem 19.7 it is proved that every bounded sequence \((s_n)\) has a convergent subsequence. Name one if \((s_n) = ((-1)^n(1 + \frac{1}{n}))\).

5. In Theorem 19.7 the claim is made that if \((s_n)\) is a sequence with finite range, then one value of the range must be attained by an infinite subsequence of \((s_n)\). Why is this?

6. In Theorem 19.7, the following argument is made: if \((s_n)\) is bounded and has infinite range, then the range must have an accumulation point \(y\). A subsequence of \((s_n)\) is constructed which converges to \(y\). Things must be at least one integer \(n_1\) such that \(s_n_1 \in N(y, 1)\). There must be at least one integer \(n_2\), larger than \(n_1\), such that \(s_n_2 \in N(y, \frac{1}{2})\). There must be at least one integer \(n_3\), larger than \(n_2\), such that \(s_n_3 \in N(y, \frac{1}{3})\). Keep on going. The subsequence \((s_{n_k})\) must converge to \(y\) because for each \(\epsilon > 0\) we have \(s_{n_k}, s_{n_k+1}, s_{n_k+2}, \cdots \in N(y, \frac{1}{\epsilon}) \subseteq N(y, \epsilon)\) (using \(k > \frac{1}{\epsilon}\)). Question: how do we know that these integers \(n_1 < n_2 < n_3 < \cdots\) can be found to make the construction work?

7. In theorem 19.8 it is proved that a sequence \((s_n)\) which has no upper bound has a subsequence \((s_{n_k})\) which converges to \(+\infty\), meaning that for all \(M > 0\) there exists \(N \in \mathbb{N}\) such that \(s_{n_k} > M\) whenever \(k > N\). The
construction is this: there must be an integer \( n_1 \) such that \( s_{n_1} > 1 \). There must be an integer \( n_2 \), larger than \( n_1 \), such that \( s_{n_2} > 2 \). There must be an integer \( n_3 \), larger than \( n_2 \), such that \( s_{n_3} > 3 \). Keep on going. What guarantees that these integers \( n_1 < n_2 < n_3 < \cdots \) can be found?

8. Let \((s_n)\) be a bounded sequence. Define the terms \( \limsup s_n \) and \( \liminf s_n \).

9. Let \((s_n)\) be a bounded sequence and let \( m = \limsup s_n \). Let \( \epsilon > 0 \) be given. Why must it be the case that only a finite number of terms in the sequence \((s_n)\) are \( \geq m + \epsilon \)? Therefore there is an \( N \in \mathbb{N} \) such that \( s_n < m + \epsilon \) whenever \( n \geq N \). An equivalent statement holds for \( n = \limsup s_n \).

10. Let \((s_n)\) be a bounded sequence and let \( m = \limsup s_n \). Then Corollary 19.12 states that \((s_n)\) must have a subsequence which converges to \( m \). The construction is this: There must be an integer \( n_1 \) so that \( m - 1 < s_{n_1} < m + 1 \). There must be an integer \( n_2 \), larger than \( n_1 \), such that \( m - \frac{1}{2} < s_{n_2} < m + \frac{1}{2} \). There must be an integer \( n_3 \), larger than \( n_2 \), such that \( m - \frac{1}{3} < s_{n_3} < m + \frac{1}{3} \). Keep on going. The subsequence \((s_{n_k})\) converges to \( m \) because for each \( \epsilon > 0 \) we have \( |s_{n_k} - m| < \frac{1}{k} < \epsilon \) whenever \( k > \frac{1}{\epsilon} \). Question: What guarantees that these integers \( n_1 < n_2 < n_3 < \cdots \) can be found?

Homework for Section 19, due ??? (only the starred problems will be graded):

1, 2, 4∗, 5∗(a, b, c), 7∗(a, b), 10∗, 13∗, 18∗

Hints:

5(a) Identify a subsequence of \((1 + \frac{1}{n})^n\) and use Exercise 18.14.

5(b) Looks like the square of a sequence.

5(c) \((1 + \frac{1}{n})^{n-1} = (1 + \frac{1}{n})^n \cdot (1 + \frac{1}{n})^{-1} \) looks like the product of two sequences.

7(a) Note that a bounded sequence has a convergent subsequence. Why does this imply that (a) is true?

7(b) Find a counterexample.

10. Consider the reciprocal sequence and use Example 19.5 and limit properties.

13. (a) Let \( p = \limsup s_n \) and \( q = \limsup t_n \). Show \( \forall \epsilon > 0 \) that \( \limsup (s_n + t_n) \leq p + q + \epsilon \). Method: By Theorem 19.11, all but a finite number of the terms of \((s_n)\) are \( \leq p + \frac{\epsilon}{2} \), and all but a finite number of the terms of \((t_n)\) are

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\[ q + \epsilon, \] so all but a finite number of the terms of \((s_n + t_n)\) are \(\leq p + q + \epsilon\) (why), so all but a finite number of terms of any convergent subsequence \((s_{n_k} + t_{n_k})\) are \(\leq p + q + \epsilon\), so every convergent subsequence has limit \(\leq p + q + \epsilon\) (why).

13(b) Use two oscillating sequences (defined on page 184 after Definition 19.9).

18. Fill in the details of the following sketch. Also, note the similarity with Problem 14.12.

Step 1: \(C\) compact implies every sequence in \(C\) has a subsequence which converges to a point in \(C\). Proof: Assume \(C\) is compact. Let \((s_n)\) be a sequence with \(s_n \in C\) for all \(n \in \mathbb{N}\). Since \(C\) is bounded (why), the sequence \((s_n)\) is bounded (why), therefore there is a convergent subsequence \((s_{n_k})\) (why) which converges to some number \(y\). We must show that \(y \in C\). Suppose not (arguing by contradiction). Since \(C\) is closed (why), we can find \(\epsilon > 0\) such that \(N(y, \epsilon) \cap C = \emptyset\). Therefore all the terms of \((s_{n_k})\) fall outside of \(N(y, \epsilon)\). This contradicts the statement that \((s_{n_k})\) converges to \(y\), because infinitely many of the terms of \((s_{n_k})\) must satisfy \(|s_n - y| < \epsilon\). Therefore \(y \in C\).

Step 2: Every sequence in \(C\) has convergent subsequence with limit in \(C\) implies \(C\) is compact. Proof: Assume that every sequence in \(C\) has convergent subsequence with limit in \(C\). It will suffice to show that \(C\) is closed and bounded (why). Suppose \(C\) is not bounded. Then we can find \(s_1 \in C\) such that \(s_1 > 1\), \(s_2 \in C\) such that \(s_2 > 2\), \(s_3 \in C\) such that \(s_3 > 3\), and so on (why). The sequence \((s_n)\) does not have a convergent subsequence (why). Contradiction. Therefore \(C\) is bounded. Now suppose \(C\) is not closed. Then there is a number \(y \notin C\) such that for all \(\epsilon > 0\) we have \(N(y, \epsilon) \cap C \neq \emptyset\) (why). Therefore we can find \(s_1 \in C\) such that \(|s_1 - y| < 1\), \(s_2 \in C\) such that \(|s_2 - y| < \frac{1}{2}\), \(s_3 \in C\) such that \(|s_3 - y| < \frac{1}{3}\), and so on (why). The sequence \((s_n)\) converges to \(y\) (why), so every subsequence converges to \(y\) (why), so no convergent subsequence of \((s_n)\) converges to a point in \(C\). Contradiction. Therefore \(C\) is closed.