Math 316-01 Intermediate Analysis

Questions for Section 17: Limit Theorems

1. In Theorem 17.1, pp. 166-167, it is proved that if \((s_n)\) converges to \(s\), \((t_n)\) converges to \(t\), \(t_n \neq 0\) for all \(n\), and \(t \neq 0\), then \((\frac{s_n}{t_n})\) converges to \(\frac{s}{t}\). To prove this, first we find \(N_1\) such that 
\[|t_n - t| < \frac{|t|}{2}\]
whenever \(n > N_1\). Then we can say 
\[|t_n| = |t - (t - t_n)| \geq |t| - |t - t_n| > |t| - \frac{|t|}{2} = \frac{|t|}{2}\]
whenever \(n > N_1\). What justifies the first inequality (\(\geq\)) in this step?

2. In the same proof, an \(N_2\) is found such that 
\[|t_n - t| \leq \frac{1}{2}|t|^2\]
whenever \(n > N_2\). This implies that 
\[\left|\frac{1}{t_n} - \frac{1}{t}\right| = \left|\frac{t - t_n}{t_n t}\right| < \frac{2}{|t|^2}|t - t_n| < \epsilon\]
whenever \(n > \max N_1, N_2\). What justifies the two inequalities (\(<\)) in this step?

3. In Example 17.6, pp. 168-169, a proof is given that if \((t_n)\) converges to \(t\) and \(t_n \geq 0\) for all \(n\) and \(t > 0\), then \((\sqrt{t_n})\) converges to \(\sqrt{t}\). The proof of the case when \(t = 0\) is left to the reader. How do you prove this case?

4. We will run through the argument of Theorem 17.7. Assume \((s_n)\) is a sequence of positive numbers and that \((\frac{s_{n+1}}{s_n})\) converges to \(L < 1\). Let \(c\) be any real number which satisfies \(L < c < 1\). What guarantees that we can find \(N\) such that \(\frac{s_{n+1}}{s_n} \leq c\) whenever \(n > N\)? Hence \(s_{n+1} \leq cs_n\) whenever \(n > N\).

5. Given \(s_{n+1} \leq cs_n\) whenever \(n > N\), why can we say that \(s_{n+k} \leq c^k s_n\) for all \(k\) whenever \(n > N\)?

6. Given \(s_{n+k} \leq c^k s_n\) whenever \(n > N\), we can say that \(s_{N+1+k} \leq c^k s_{N+1}\) for all \(k \in \mathbb{N}\). Given that we know that \((c^k)\) converges to 0 (it must because \(0 < c < 1\)), why does this imply that \((S_k)\) converges to 0, where \(S_k = s_{N+1+k}\)? Hence \((s_n)\) converges to 0.

7. Define what it means for \((s_n)\) to diverge to \(+\infty\).

8. In Example 17.11 on page 170 it is proved that \(\left(\frac{4n^2 - 3}{n+2}\right)\) diverges to \(+\infty\). We will give an alternative proof. Complete the following steps: 
\[\frac{4n^2 - 3}{n+2} = \]
\[
\frac{n^2 4 - 3/n^2}{n 1 + 2/n} = n^{4 - 3/n^2}. \text{ Since } \left(\frac{4 - 3/n^2}{1 + 2/n}\right) \text{ converges to } 4, \text{ there must be an } N \text{ so that } \frac{4 - 3/n^2}{1 + 2/n} > 3 \text{ whenever } n > N, \text{ therefore } \frac{4n^2 - 3}{n+2} > 3n \text{ whenever } n > N, \text{ therefore } \left(\frac{4n^2 - 3}{n+2}\right) \text{ diverges to } +\infty.
\]

Homework for Section 17, due ??? (only the starred problems will be graded):

1, 2, 3\textsuperscript{*}(a, b), 4\textsuperscript{*}(a, b), 5\textsuperscript{*}(l), 8\textsuperscript{*}, 10\textsuperscript{*}, 15\textsuperscript{*}(c)

Hints:

5(l). Use Theorem 17.7.

15(c). Start with

\[
\sqrt{n^2 + n} - n = \frac{(\sqrt{n^2 + n} - n)(\sqrt{n^2 + n} + n)}{\sqrt{n^2 + n} + n} = \frac{n}{\sqrt{n^2 + n} + n} = \frac{1}{\sqrt{1 + 1/n} + 1}
\]

and go from there.