Math 316-01 Intermediate Analysis

Questions for Section 14: Compact Sets

1. For each \( n \in \mathbb{N} \) let \( A_n = (n - \frac{1}{n}, n + \frac{1}{n}) \). Why does the collection of sets \( F = \{A_n : n \in \mathbb{Z}\} \) form an open cover of \( \mathbb{N} \)?

2. Let \( G = \{A_{n_1}, A_{n_2}, \ldots, A_{n_k}\} \) be a finite subcover of \( F \) as in Question 1, where \( n_1 < n_2 < \cdots < n_k \). Why isn’t \( G \) an open cover of \( \mathbb{N} \)?

3. Given the information in Questions 1 and 2, why is \( \mathbb{N} \) not compact?

4. For each \( x \in \mathbb{Z} \) let \( A_x = (-\infty, x) \cup (x, \infty) \). Why does the collection of sets \( F = \{A_x : x \in \mathbb{Z}\} \) form an open cover of \( \mathbb{R} \)?

5. Let \( G = \{A_0, A_1\} \) be one of the many finite subcovers of \( F \) as in Question 4. Explain why \( G \) is an open cover of \( \mathbb{R} \).

6. Explain why the existence of \( G \) in Question 5 does not prove that \( \mathbb{R} \) is compact.

7. In Lemma 14.4 (page 139) it is proved that a non-empty closed bounded subset of \( \mathbb{R} \) must have a maximum element. If we remove the hypothesis that the subset must be closed, then the conclusion of the lemma may be false. Give an example of a non-empty bounded subset of \( \mathbb{R} \) which does not have a maximum element.

Remark: Questions 8 through 14 deal with the proof of the Heine-Borel Theorem (Theorem 14.5)

8. Let \( S \subseteq \mathbb{R} \) be compact. An open cover of \( S \) is \( F = \{(-n, n) : n \in \mathbb{N}\} \). Since \( S \) is compact, \( S \) must be covered by a finite number of the intervals in \( F \). Why does this imply that \( S \) is bounded?

9. Let \( S \subseteq \mathbb{R} \) be compact. To show that \( S \) is closed, we must show that for each \( p \notin S \) we can find \( \epsilon > 0 \) such that \( N(p, \epsilon) \cap S = \emptyset \). To this end, explain why the family of sets \( F = \{(-\infty, p - \frac{1}{n}) \cup (p + \frac{1}{n}, \infty) : n \in \mathbb{N}\} \) forms an open cover of \( S \). Note that the book uses the notation \( U_n = (-\infty, p - \frac{1}{n}) \cup (p + \frac{1}{n}, \infty) \) (see Figure 14.2, page 140 for some examples).

10. Say that \( G = \{U_{n_1}, U_{n_2}, \ldots, U_{n_k}\} \) is an open cover of the set \( S \) in Question 9, where \( n_1 < n_2 < \cdots < n_k \). Why do we have \( S \subseteq (-\infty, p - \frac{1}{n_k}) \cup (p + \frac{1}{n_k}, \infty) \)?

11. Give the information in Question 10, why do we have \( N(p, \frac{1}{n_k}) \cap S = \emptyset \)? Hence \( S \) is closed.

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12. Now assume that $S$ is closed and bounded. We will show that $S$ must be compact. So let $\mathcal{F}$ be an arbitrary open cover of $S$. For concreteness we will write this as $\mathcal{F} = \{F_i : i \in I\}$, where $I$ is some index set, and we have $S \subseteq \bigcup_{i \in I} F_i$. We would like to show that $S$ is covered by a finite number of the sets in $\mathcal{F}$. The book defines $S_x = S \cap (-\infty, x]$ for each $x \in \mathbb{R}$ (see Figure 14.3, page 140). Some of these $S_x$ are covered by a finite number of sets in $\mathcal{F}$. For example, if $d$ is the minimum element of $S$ (which exists since $S$ is closed and bounded by Lemma 14.4), then $S_d = \{d\}$, and clearly $S_d$ can be covered by just one set from $\mathcal{F}$. Let $B$ be the set of all real numbers $x$ such that $S_x$ can be covered by a finite number of sets from $\mathcal{F}$. The book suggests that if $B$ has no upper bound then $S$ can be covered by a finite number of sets from $\mathcal{F}$. What justifies this claim?

13. Given Question 12, our task now is to show that $B$ has no upper bound. This is proved by contradiction. So assume $m$ is the least upper bound of $B$. The contradiction we are seeking is $m \neq \sup B$. There are two cases to consider: $m \in S$, or $m \not\in S$ (we don’t know which of these is true in general, so we treat both cases). First suppose $m \in S$. How do we know that $m \in F_0$ for some $F_0 \in \mathcal{F}$? Why does this imply that $[x_1, x_2] \subseteq F_0$ for some pair of real numbers $x_1, x_2$ such that $x_1 < m < x_2$? How do we know that $S_{x_1}$ is covered by a finite number of sets in $\mathcal{F}$? How do we know that therefore $S_{x_2}$ is also covered by a finite number of sets in $\mathcal{F}$? Why does this contradict the definition of $m$?

14. We now consider $m \not\in S$, still assuming that $m$ is the least upper bound of all $x$ such that $S_x$ is covered by a finite number of sets in $\mathcal{F}$. Why does this imply that $(m - \epsilon, m + \epsilon) \cap S = \emptyset$ for some $\epsilon > 0$? Why does this imply that $S_{m-\epsilon} = S_{m+\epsilon}$? How do we know $m - \epsilon \in B$? How do we know $m + \epsilon \in B$? Why does this contradict the definition of $m$? Therefore $B$ has no upper bound. We can pick $b \in B$ to be larger than the least upper bound of $S$. Since $S_b$ is covered by a finite number of sets in $\mathcal{F}$, and since $S = S_b$, therefore $S$ can be covered by a finite number of sets.

15. Why is the set $[0, 1] \cup [2, 3]$ compact?
Remark: Questions 16 through 20 deal with the proof of the Bolzano-Weierstrass Theorem (Theorem 14.6).

16. The Bolzano-Weierstrass theorem states the following:

For all bounded subsets $S$ of $\mathbb{R}$, if $S$ is infinite then $S' \neq \emptyset$.

What is the the contrapositive of this statement?

17. Let $S$ be a bounded subset of real numbers which has no accumulation points. Why is $S$ closed? Hence $S$ is closed and bounded, hence compact.

18. Given $S$ bounded such that $S' = \emptyset$, why can we say that each $x \in S$ there exists an $\epsilon_x > 0$ such that $N(x, \epsilon_x) \cap S = \{x\}$?

19. Why is $\mathcal{F} = \{N(x, \epsilon_x) : x \in S\}$ an open cover of $S$?

20. Since $S$ is compact, it is covered by a finite number of sets in $\mathcal{F}$. Why does this imply that $S$ is a finite set?

21. By the Bolzano-Weierstrass Theorem, the set $S = \{(-1)^n(1 + \frac{1}{n}) : n \in \mathbb{N}\}$ has an accumulation point. Name one of them.

22. The converse to the Bolzano-Weierstrass Theorem states that if $S$ is bounded and has no accumulation points then $S$ must be infinite. Why is the set $S = \{1, 2, 3\}$ a counter-example to this statement?

Remark: Questions 23 through ??? deal with the proof of Theorem 14.7.

23. Theorem 14.7 states that if every finite intersection of a family of compact sets is non-empty, then the intersection of the entire family is non-empty. What is the contrapositive of this statement?

24. Let $\mathcal{F} = \{K_i : i \in I\}$ be a family of compact sets with $\bigcap_{i \in I} K_i = \emptyset$. We will show that there is a finite intersection of these sets which is empty. To begin with, why is each complement set $\mathbb{R} \setminus K_i$ open? Why is the family $\mathcal{F}' = \{\mathbb{R} \setminus K_i : i \in I\}$ an open cover of $\mathbb{R}$?

25. Let $K_{i_0}$ be chosen in $\mathcal{F}$ at random. Why is $K_{i_0}$ covered by a finite number of the sets in $\mathcal{F}'$?

26. Say that $K_{i_0} \subseteq (\mathbb{R} \setminus K_{i_1}) \cup \cdots \cup (\mathbb{R} \setminus K_{i_n})$. Why can we say that $K_{i_0} \subseteq \mathbb{R} \setminus (K_{i_1} \cap \cdots \cap K_{i_n})$? Why does this imply that $K_{i_0} \cap K_{i_1} \cap \cdots \cap K_{i_n} = \emptyset$?
Homework for Section 14, due ??? (only the starred problems will be graded):

1, 2, 3*, 4*, 5*, 6*, 8*, 12*

Hints:

3(d). Let $S = \{ r \in \mathbb{Q} : r > 0$ and $r^2 < 2 \}$. For each $r \in S$ let $A_r = (-r, r) \cup (\sqrt{2}, 2 + r)$. Set $\mathcal{F} = \{ A_r : r \in S \}$. Explain why $\mathcal{F}$ is an open cover of $\{ x \in \mathbb{Q} : 0 \leq x \leq 2 \}$. Explain why $\{ x \in \mathbb{Q} : 0 \leq x \leq 2 \}$ cannot be covered by a finite number of the sets in $\mathcal{F}$. See page 124.

4. Let $\mathcal{C} = \{ C_i : i \in I \}$ be a collection of compact subsets of $\mathbb{R}$. Explain why $\bigcap_{i \in C} C_i$ is closed and bounded, and why this proves that the intersection is compact.

5(a). Let $\mathcal{F}$ be an open cover of $S_1 \cup S_2$. Explain why $\mathcal{F}$ must be an open cover of both $S_1$ and $S_2$. Therefore a finite subcover $\mathcal{G}_1$ covers $S_1$, and another finite subcover $\mathcal{G}_2$ covers $S_2$. How do you construct a finite subcover of $S_1 \cup S_2$ from these?

5(b). Use the Heine-Borel theorem to construct a non-compact set out of an infinite union of compact sets.

8(a). Let $\mathcal{F} = \{ A_i : i \in I \}$ be an open cover of $T$. Explain why $\mathcal{F}' = \{ A_i \cup (\mathbb{R}\setminus T) : i \in I \}$ is an open cover of $S$. Therefore $S$ is covered by $A_{n_1} \cup (\mathbb{R}\setminus T), \ldots, A_{n_k} \cup (\mathbb{R}\setminus T)$. Explain why this implies that $T$ is covered by $A_{n_1}, \ldots, A_{n_k}$.

12. We must prove that if $S \subseteq \mathbb{R}$ is compact, then every infinite subset $T$ of $S$ has an accumulation point in $S$. We must also prove the converse of this statement. Follow this outline:

Step 1. If $S$ is compact, then it is closed and bounded (why?), so every infinite subset of $S$ is bounded (why?), therefore every infinite subset $T$ of $S$ has an accumulation point $x$ (why?), which makes $x$ an accumulation point of $S$ (why?), and therefore $x$ must belong to $S$ (why?)

Step 2. The converse says that if every infinite subset of $S$ has an accumulation point, then $S$ is compact. This is equivalent to proving the converse, namely if $S$ is not compact, then it has an infinite subset (call it $T$) which does not have an accumulation point in $S$. Now assume that $S$ is not compact. Then it is not closed or it is not bounded (why?). If it is not closed, then there must be some $x$ outside of $S$ such that $N(x, \epsilon) \cap S \neq \emptyset$ for every
\( \epsilon > 0 \) (why?). So it is possible to construct (how?) a sequence of numbers \( x_1, x_2, x_3, \ldots \) with the following properties: All of these numbers belong to \( S \), and \(|x - x_1| > |x - x_2| > |x - x_3| > \cdots \). Therefore \( T = \{x_1, x_2, x_3, \cdots \} \) is an infinite subset of \( S \), the only accumulation point of \( T \) is \( x \), but \( x \notin S \). On the other hand, if \( S \) is not bounded, then it is possible to construct (how?) a sequence of numbers \( x_0, x_1, x_2, x_3, \ldots \) belonging to \( S \) such that \(|x_1 - x_0| > 1, |x_2 - x_1| > 2, |x_3 - x_2| > 3, \ldots \). Setting \( T = \{x_0, x_1, x_2, \ldots \} \), we have an infinite subset of \( S \) with no accumulation points at all.