Math 316-01 Intermediate Analysis

Questions for Section 13: Topology of the Reals

1. Express the neighborhood $N(x, \varepsilon)$ in interval notation. Find a neighborhood of $-3$ in $[-5, -2) \cup (-2, 0) \cup \mathbb{N}$.

2. Express the neighborhood $N^*(x, \varepsilon)$ in interval notation. Find a deleted neighborhood of $-3$ in $[-5, -2) \cup (-2, 0) \cup \mathbb{N}$.

3. Define interior point of a set. Find all interior points in $[-5, -2) \cup (-2, 0) \cup \mathbb{N}$.

4. Define boundary point of a set. Find all boundary points of $[-5, -2) \cup (2, 0) \cup \mathbb{N}$.

5. Define accumulation point of a set. Find all accumulation points of $[-5, -2) \cup (-2, 0) \cup \mathbb{N}$.

6. What’s the difference between a boundary point and an accumulation point of a set? Give an example.

7. Define open set. Is $[-5, -2) \cup (-2, 0) \cup \mathbb{N}$ open? Why or why not?

8. Define closed set. Is $[-5, -2) \cup (-2, 0) \cup \mathbb{N}$ closed? Why or why not?

9. Define the closure of a set. What is the closure of $[-5, -2) \cup (-2, 0) \cup \mathbb{N}$?

Remark: Questions 10 through 16 constitute a proof of Theorem 13.17.

10. The negation of the statement If $S$ is closed then $S' \subseteq S$ is the statement $S$ is closed and $\exists x : x \in S' \land x \notin S$. Why is the negated statement false?

11. Assume that $S' \subseteq S$. Let $x \notin S$ be given. Why does there exist $\varepsilon > 0$ such that $N(x, \varepsilon) \cap S = \emptyset$? Why does this prove that $S$ is closed?

12. Why do the information in Questions 10 and 11 constitute a proof that $S$ is closed iff $S' \subseteq S$?

13. In order to show that $\text{cl} \ S$ is closed, we must show $(\text{cl} \ S)' \subseteq \text{cl} \ S$ by Part (a) of Theorem 13.17, which we have just proved in Questions 10 – 12. Let $x \in (\text{cl} \ S)'$ be given. We must show $x \in \text{cl} \ S$. In other words, we must show every $N^*(x, \varepsilon)$ contains an $s \in S$. Let $N^*(x, \varepsilon)$ be given. Why does $N^*(x, \varepsilon)$ contain a $y \in \text{cl} \ S$?
14. (Continuation) Consider the \( y \in \text{cl} \, S \) we found in Question 13. We know it belongs to \( N^*(x, \epsilon) \). Why is it true that there is a \( \delta > 0 \) such that \( N^*(y, \delta) \subseteq N^*(x, \epsilon) \)?

15. (Continuation) Why is it true that the \( N^*(y, \delta) \) we found in Question 14 contains an \( s \in S \)?

16. (Continuation) Why is it true that the \( s \in S \) we found in Question 15 belongs to \( N^*(x, \epsilon) \)?
Homework for Section 13, due ??? (only the starred problems will be graded):

2, 3(d,e), 4*(d,e), 5*(c,f), 6*(c,f), 7*(a,b,e), 20*(c,d), 21*(d,e)

Hints:

5(c). Interpret Theorems 12.12 and 12.14 to mean that strictly between every two real numbers \( x \) and \( y \) there is a rational number \( p \) and an irrational number \( q \). Explain why these theorems imply that \( \mathbb{Q} \) is neither open nor closed.

6(c). Use Theorem 12.12 to find the accumulation points of \( \mathbb{Q} \).

7(a) Explain why \( S = \{ \frac{1}{n} : n \in \mathbb{N} \} \) is a counterexample.

7(b) Explain why \( S = \mathbb{Z} \) is a counterexample.

7(e) Explain why \( S = \mathbb{Q} \) is a counterexample. Use Theorems 12.12 and 12.14 to compute the boundary points of \( S \).

20(c) Here is an outline of the proof. Fill in the details. Let \( x \in \text{cl} (S \cap T) \) be given. We must show that \( x \in \text{cl} S \) and \( x \in \text{cl} T \). There are two cases: \( x \in S \cap T, x \in (S \cap T)' \). If \( x \in S \cap T \), explain why \( x \in \text{cl} S \) and \( x \in \text{cl} T \). If \( x \in (S \cap T)' \), explain why every punctured neighborhood of \( x \) intersects \( S \cap T \). Then explain why every punctured neighborhood of \( x \) intersects both \( S \) and \( T \). Then explain why \( x \in \text{cl} S \) and \( x \in \text{cl} T \).

20(d) Explain why \( S = [0,1) \cup (1,2] \) and \( T = \{1\} \) is a counterexample.

21(d) Here is an outline of the proof. Fill in the details. Let \( x \in (\text{int} S) \cup (\text{int} T) \) be given. We must show \( x \in \text{int} (S \cup T) \). There are two cases: \( x \in \text{int} S, x \in \text{int} T \). If \( x \in \text{int} S \), then \( N(x, \epsilon) \subseteq S \) for some \( \epsilon \). Explain why this implies \( x \in \text{int} (S \cup T) \). If \( x \in \text{int} T \), then explain why \( x \in \text{int} (S \cup T) \).

21(e) Explain why \( S = \mathbb{Z} \) and \( T = \mathbb{R} \setminus \mathbb{Z} \) is a counterexample.