Math 316-01 Intermediate Analysis
Questions for Section 12: The Completeness Axiom

1. Name two different upper bounds for the set $[0, 3)$
2. Identify the least upper bound of the set $[0, 3)$
3. Explain why
   \[ a \leq b \Rightarrow (\forall \epsilon > 0 : a < b + \epsilon) \]
   is a true statement.
4. Explain why
   \[ a > b \Rightarrow (\exists \epsilon > 0 : a \geq b + \epsilon) \]
   is a true statement.
5. Given that the statement in Question 4 is true, explain why
   \[ (\forall \epsilon > 0 : a < b + \epsilon) \Rightarrow a \leq b \]
   is a true statement.
6. Given the true of the statements in Question 3 and Question 5, explain why
   \[ a \leq b \iff (\forall \epsilon > 0 : a < b + \epsilon) \]
   is a true statement.
7. What does the Completeness Axiom say?

Remark: Questions 8 through 13 below flesh out the proof of Theorem 12.7.
8. Let $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$ be given. Let $C = \{x + y : x \in A, y \in B\}$ be the set of all possible sums using numbers in $A$ and $B$. Let $X$ be any upper bound for $A$ and let $Y$ be any upper bound for $B$. Why is $X + Y$ an upper bound for $C$?
9. (Continuation of 8) Why is $\text{sup } C \leq X + Y$ true?
10. (Continuation) Why does this imply that $\text{sup } C \leq \text{sup } A + \text{sup } B$?
11. (Continuation) In light of Question 10, to prove that $\text{sub } C = \text{sup } A + \text{sup } B$ it suffices to prove that $\text{sup } A + \text{sup } B \leq \text{sup } C$. Explain why this is equivalent to proving the truth of the statement
   \[ \forall \epsilon > 0 : \text{sup } A + \text{sup } B < \text{sup } C + \epsilon. \]
12. (Continuation) Let $\epsilon > 0$ be given. Explain why there exists an $x \in A$ such that

$$\sup A - \frac{\epsilon}{2} < x$$

and why there exists a $y \in B$ such that

$$\sup B - \frac{\epsilon}{2} < y.$$ 

13. (Continuation) Let $\epsilon > 0$ be given. Explain why there exists a $c \in C$ such that

$$\sup A + \sup B - \epsilon < c.$$ 

Why does this imply $\sup A + \sup B < \sup C + \epsilon$?

**Remark:** Questions 14 through 16 below flesh out the proof of Theorem 12.11.

14. On page 124, the following argument is made: if $x^2 < p$ then there exists an $n \in \mathbb{N}$ such that $(x + \frac{1}{n})^2 < p$. Explain why this proves that

$$x \neq \sup \{r \in \mathbb{R} : r^2 < p\}.$$ 

15. On page 124, the following argument is made: if $x^2 > p$ then there exists an $m \in \mathbb{N}$ such that $(x - \frac{1}{m})^2 > p$. Explain why this proves that

$$x \neq \sup \{r \in \mathbb{R} : r^2 < p\}.$$ 

16. Let $x = \sup \{r \in \mathbb{R} : r^2 < p\}$. In light of Questions 14 and 15, why is it true that $x^2 = p$?

17. Let $x = \sup \{r \in \mathbb{Q} : r^2 < p\}$. In light of Questions 14 through 16, why is it true that $x^2 = p$?
Homework for Section 12, due ??? (only the starred problems will be graded):

3*(g, h), 6, 7*, 8, 9, 12*

Hints:

7. I will prove the first part of (a) as a template for the second half and for part (b). Let \( k \geq 0 \) be given. If \( k = 0 \) then clearly the statements in (a) are true, so we will assume \( k > 0 \). We have \( \sup S \geq x \) for all \( x \in S \), therefore \( k \cdot \sup S \geq kx \) for all \( kx \in kS \), therefore \( k \cdot \sup S \) is an upper bound of \( kS \), therefore \( k \cdot \sup S \geq \sup (kS) \). On the other hand, we know that \( \sup (kS) \geq kx \) for all \( x \in S \), therefore \( \frac{\sup (kS)}{k} \geq x \) for all \( x \in S \), therefore \( \frac{\sup (kS)}{k} \) is an upper bound for \( S \), therefore \( \frac{\sup (kS)}{k} \geq \sup S \), therefore \( \sup (kS) \geq k \cdot \sup S \). The two inequalities \( k \cdot \sup S \geq \sup (kS) \) and \( \sup (kS) \geq k \cdot \sup S \) imply \( \sup (kS) = k \cdot \sup S \).

9.(a) Review the statement of the well-ordering property. What theorem in Section 12 guarantees that the set \( \{ m \in \mathbb{N} : m > y \} \) is non-empty?

12. (a) \( f + g \) is notation for the function which has input \( x \) and output \( f(x) + g(x) \). Let \( M \) be an upper bound for \( f(D) \) and \( N \) be an upper bound for \( g(D) \). Use it find an upper bound for \( (f+g)(D) \), then prove the inequality.

12. (b) Drawing possible graphs will help produce a particular \( f \) and \( g \).