Questions for Section 11: Ordered Fields

1. State the trichotomy law for the two real numbers $x$ and $y$.

2. Let $x$ and $y$ be real numbers. Assume that $x \leq y$ and $y \leq x$. Prove, using the trichotomy law, that $x = y$.

3. Let $x$ be a real number. How does the book define the symbol $-x$? (The answer is contained on one of the field axioms.)

4. Using the definition of $-x$ as above, prove that $-0 = 0$ using combining two of the field axioms.

5. Assume that $x > 0$. Prove that $-x < 0$ using $-0 = 0$ and Theorem 1.1.

6. State the triangle inequality for two real numbers $x$ and $y$.

7. Note that the textbook discussion of Section 11 begins with an assumption that there are two operations called addition (+) and multiplication (·), but never defines exactly what they are. We will assume that they have their usual meanings. Note also that the book makes no reference to subtraction (−), but does refer to the existence of the symbol $-x$. We will agree to define $x - y$ as $x + (-y)$. Given this, prove that $|x - y| \leq |x| + |y|$ using the triangle inequality and Part (c) of Theorem 11.9.

8. Prove that $x > y$ if and only if $x - y > 0$ using axiom $O3$ and $x - y = x + (-y)$.

9. Prove that $x < y$ if and only if $x - y < 0$ using the previous problem and Part (e) of Theorem 11.1.

10. Prove that $|x-y| = |y-x|$ using the trichotomy law for $x$ and $y$ combined with the Questions 8 and 8 above.

11. Prove that $|x| \leq |x-y| + |y|$ for any $x$ and $y$ using the triangle inequality.

12. Prove that $|y| \leq |x| + |x-y|$ for any $x$ and $y$ using the triangle inequality.
Homework for Section 11, due ??? (only the starred problems will be graded):

1, 3*(d, g, h), 6*, 7*

Hints:

3. Restrict all the steps in your proof to either axioms or theorems in this section, including the facts established above.

6. First prove that $-|x - y| \leq |x| - |y| \leq |x - y|$ using Questions 10, 11, and 12 above, then use Part (b) of Theorem 11.9 to finish the proof.

7. Prove this by induction on $n \geq 2$ using the triangle inequality.