Give complete proofs for all problems.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be given by
   
   
   $$f(x) = \begin{cases} 
   x^2 \sin x & x \in \mathbb{Q} \\
   x^3 \cos x & x \not\in \mathbb{Q}. 
   \end{cases}$$

   Give an $\epsilon$-$\delta$ proof that $f'(0) = 0$.

2. Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$. Assume that $f(a) = a^2$ and $f(b) = b^2$. For example, $a = 0$, $b = 1$, $f(x) = x^5 - 4x^4$, but there are many other possibilities. Prove that there exists $c \in (a, b)$ such that $f''(c) = 2c$. Hint: consider the function $f : [a, b] \to \mathbb{R}$ defined by $g(x) = f(x) - x^2$.

3. Let $f : [0, \frac{1}{2}] \to \mathbb{R}$ be defined by $f(x) = \ln(1 - x)$.
   
   (a) Use the $3^{rd}$ Taylor polynomial associated with $f$ at $x_0 = 0$ to approximate the value of $\ln(\frac{1}{2})$. You can assume that $\ln(1 - x)$ is differentiable on $[0, \frac{1}{2}]$. Make sure to verify the hypotheses of Taylor’s Theorem.

   (b) Find an upper bound for the absolute value of the error of the estimate in Part (a) without a calculator. Justify your answer.

4. Let $f : [0, 10] \to \mathbb{R}$ be defined by
   
   $$f(x) = \begin{cases} 
   500 & 0 \leq x \leq 5 \\
   1000 & 5 < x \leq 10. 
   \end{cases}$$

   Let $\epsilon > 0$ be given. Find a partition $P$ of $[0, 10]$ such that $U(f, P) - L(f, P) < \epsilon$. Hint: Use $P = \{x_0, x_1, x_2, x_3\}$ for an appropriate choice of $x_0, x_1, x_2, x_3$.

5. Compute the improper integral $\int_0^\pi \frac{x \cos x - \sin x}{x^2} \, dx$. Justify the steps in the calculation, verifying that all the hypotheses of any theorem you cite are present. You can assume that $\frac{\sin x}{x}$ is differentiable on $(0, \infty)$ and has derivative function equal to $\frac{x \cos x - \sin x}{x^2}$ and that the derivative function is continuous on $(0, \infty)$.