Chapter 5: Connectivity

Section 5.1: Vertex- and Edge-Connectivity

Let $G$ be a connected graph. We want to measure how connected $G$ is.

Vertex cut: $V_0 \subseteq V$ such that $G - V_0$ is not connected

Edge cut: $E_0 \subseteq E$ such that $G - E_0$ is not connected

Vertex connectivity of connected graph: $\kappa_v(G) =$ minimum size $V_0$ such that $G - V_0$ is disconnected or a single vertex. (We will say disconnected to mean either). $\kappa_v(K_n) = n - 1$.

If $\kappa_v(G) \geq k$ then you must delete at least $k$ vertices to disconnect $G$. $G$ is $k$-vertex-connected then it has this property. Another way to say this: result is connected when you delete any $k - 1$ vertices.

Edge connectivity of connected graph: $\kappa_e(G) =$ minimum size $E_0$ such that $G - E_0$ is disconnected.

If $\kappa_e(G) \geq k$ then you must delete at least $k$ edges to disconnect $G$. Result is connected when you delete any $k - 1$ edges. $G$ is $k$-edge-connected then it has this property.

$\kappa_e(G) \leq \delta$. Reason: you can disconnect graph by deleting $\delta$ edges. So the minimum number is $\leq \delta$.

Partition cut: an edge cut with the following property: you can 2-color the edges in the cut.

**Proposition 5.1.2:** A graph $G$ is $k$-connected if and only if every partition-cut contains at least $k$ edges.

**Proof:** Assume $G$ is $k$-connected. Then no edge cut contains $k - 1$ edges, in particular any partition-cut. So every partition-cut contains at least $k$ edges.

Conversely, suppose every partition-cut contains at least $k$ edges. Suppose that $G - E_0$ is not connected. Then there is a subset $E_1 \subseteq E_0$ in which $G - E_1$ has exactly two components. Color the vertices in one of the components 0, and color the vertices in the other components 1. Let $P$ be the edges in $G$ which have edges with opposite color endpoints. Claim: $P \subseteq E_1$. Reason: If there is an edge in $P$ but not in $E_1$, it would connect the two components of $G - E_1$, which is impossible. Therefore $|E_0| \geq |E_1| \geq |P| \geq k$. Hence $G$ is $k$-connected.

**Theorem:** $\kappa_e(G) \geq \kappa_v(G)$.

**Proof:** By induction on $\kappa_e(G)$. If $\kappa_e(G) = 1$, then $G$ has a bridge edge. If there are exactly 2 vertices, then deleting a vertex reduces graph to single vertex. If $\geq 3$ vertices, then deleting the right vertex will disconnect graph. Hence $\kappa_v(G) = 1$.

Now consider $\kappa_e(G) = k$. Pick any edge $e$ in a cut of size $k$. Then $\kappa_e(G - e) \leq k - 1$. Hence $\kappa_v(G - e) \leq k - 1$. Therefore by the induction hypothesis it is possible to disconnect $G - e$ by removing $\leq k - 1$ vertices. If in removing $V_{k-1}$ we also removed $e$, then removing $V_{k-1}$
disconnects \( G \). If we did not remove \( e \), then \( e \) is a bridge edge of \( G - V_{k-1} \). If just two vertices in \( G - V_{k-1} \), then removing another vertex takes to one vertex, which disconnects \( G \). If more than two vertices, use the same argument as above to remove another vertex. Hence we can disconnect \( G \) with \( k \) vertices. Hence \( \kappa_v(G) \leq k = \kappa_e(G) \).

**Definition:** two paths from \( u \) to \( v \) \((u \neq v)\) are internally disjoint if they have no common internal vertex. They can be glued together to form a cycle.

**Theorem:** Let \( G \) be connected with 3 or more vertices. \( G \) is 2-connected iff for every pair \( u \neq v \) there are two internally disjoint paths between them.

**Proof:** Assume the condition on paths holds. We must show that no vertex disconnects graph. Cut \( x \), and consider \( u \neq v \) in \( G - x \).

There are two internally disjoint paths from \( u \) to \( v \) in \( G - x \). \( x \) cannot be internal to both. Therefore one of them survives in \( G - x \), and this supplies a path between them in \( G - x \).

Conversely, assume \( G \) is 2-connected, and let \( u \neq v \) be given. Suppose every pair of distinct paths between then share an internal vertex somewhere. We will construct internally disjoint paths by induction on the distance between \( u \) and \( v \).

**distance = 1:** show they belong to a cycle.

\( \kappa_v(G) \geq \kappa(G) = 2 \), hence cutting one edge doesn’t disconnect graph. Cut \( uv \) edge. There is still path from \( u \) to \( v \). This is internally disjoint to the \( uv \) edge.

Now assume this can be done for distance \(< k \). Consider \( d(u, v) = k \). Let \( d(u, w) = k - 1 \), where \( w \) and \( v \) are connected by an edge. There are internally disjoint paths between \( u \) and \( w \) by induction hypothesis. Call them \( P \) and \( Q \). Let \( R \) be \( uv \) path in \( G - w \).

**Case 1:** \( R \) never intersects \( P \) or \( Q \). Use \( R \) and \( P + wv \).

**Case 2:** \( R \) intersects \( P \) or \( Q \). Let \( z \) be the last vertex it hits, without loss of generality in \( P \). First path: \( P \) from \( u \) to \( z \) then \( R \) from \( z \) to \( v \). Second path: \( Q \) from \( u \) to \( w \) then \( wz \).

**NOTE:** typo in problem 5.1.24.

**Expansion Lemma:** If \( G \) is \( k \)-connected, and we add a new vertex \( V \) with edges to \( k \) existing vertices and call this \( G' \), then \( G' \) is also \( k \)-connected.

**Proof:** Cut \( k - 1 \) vertices. We must show that \( G' - V_{k-1} \) is still connected. We know that \( G - V_{k-1} \) is still connected. If \( V \) was deleted, then \( G' - V_{k-1} = G - V_{k-1} \). But if \( V \) was not deleted, there is surviving edge from \( V \) to \( G \), so there are paths from \( V \) to every other vertex in \( G' - V_{k-1} \).

**Theorem:** Let \( G \) be graph with at least 3 vertices. FAE:

1. \( G \) is connected and has no cut-vertex.
2. \( G \) is 2-connected.
3. Every pair \( u \neq v \) has 2 internally disjoint paths.
4. Every pair \( u \neq v \) belongs to cycle.
5. There are no isolated vertices, and every pair of edges belongs to a cycle.

**Proof:**

1 through 4 are equivalent.

4 implies 5: Assume every pair of vertices belongs to a cycle. This implies 1, so we can assume we have all the properties of 1 through 4. Clearly there are no isolated vertices. Let $uv$ and $xy$ be two edges. If they share and endpoint, we can delete the common endpoint and there will still be a path between the two other endpoints. Hence these edges belong to a cycle.

Now suppose $uv$ and $xy$ do not share an endpoint. Create new vertices $W$ and $Z$ and add them to $G$, connecting $W$ to $u, v$ and $Z$ to $x, y$. By expansion lemma, $G'$ is 2-connected. Therefore $W$ and $Z$ belong to cycle of $G'$. This can be contracted to cycle through the two edges.

5 implies 1: First show connected. Let $u$ and $v$ be given. If $uv$ edge, they are connected by path. If no $uv$ edge, they belong to two edges which belong to cycle, hence they are connected by path.

Next show no cut-vertex. Let $x$ be vertex in graph. Cut $x$. We must show $G - x$ is connected.

Let $a, b$ be two vertices left over. If no edge, they belong to a cycle in $G$ by the argument above, hence to two internally disjoint paths in $G$. One of these survives in $G - x$.

**Section 5.2: Constructing Reliable Networks**

**Lemma 5.2.1:** If you add a path to a 2-connected graph you get a 2-connected graph.

**Proof:** It is still true that every pair of vertices lies on a cycle.

**Theorem 5.2.2:** $G$ is 2-connected if and only if it can be obtained by adding paths to a cycle.

**Proof:** Sufficiency is clear by the lemma. Now assume $G$ is 2-connected. We will construct $G$ by adding paths to a cycle as follows:

Since $G$ is 2-connected, it contains a cycle. Therefore it has a subgraph which can be generated by an ear decomposition. We will now show that any proper ear-decomposition subgraph $H$ can be enlarged to a larger one $H'$. Eventually we arrive at $G$.

Let $uv$ be an edge in $G$ which does not belong to $H$. If both $u$ and $v$ are vertices in $H$, then the new ear-decomposition subgraph is $H + uv$ and we are done. Otherwise, let $xy$ be any edge in $H$. Then $uv$ and $xy$ belong to a cycle $C$ of $G$ since $G$ is 2-connected. Since either $u$ or $v$ does not belong to $H$, we can use $C$ to construct a new ear for $H$.

**Lemma 5.2.3:** If you add a path or a cycle to a 2-edge-connected graph you get a 2-edge-connected graph.

**Proof:** If no bridges to begin with, then no bridges after the addition.
Theorem 5.2.2: \( G \) is 2-edge-connected if and only if it can be obtained by adding paths or cycles to a cycle.

Proof: Sufficiency is clear by the lemma.

Conversely, let \( G \) be any 2-edge-connected graph. \( G \) contains cycle. We will show that we can continue to extend it by paths and cycles until we get to \( G \).

Let \( H \) be subgraph of \( G \) which results from cycle by adding paths and cycles. Suppose \( H \) contains all vertices of \( G \). Then any \( uv \) edge not in \( H \) can be added to \( H \), creating larger thingy. If on the other hand \( G \) contains vertex not in \( H \), then by connectedness there is a frontier edge from vertex in \( H \) to vertex not in \( H \). It cannot be a bridge edge of \( G \), so it lives in cycle of \( G \). Follow the cycle both directions until it hooks up with \( H \). If it hooks up with only one vertex of \( H \), it is a closed ear. Otherwise it is an open ear. Add to \( H \), producing \( H' \).

See Theorem 5.2.5 for characterizations of 3-connected graphs. See Theorem 5.3.6 for a characterization of \( k \)-connected graphs.

Proposition 5.2.6: A \( k \)-connected graph on \( n \) vertices has at least \( \frac{kn}{2} \) edges.

Proof: We have \( 2e \geq n\delta \geq nk \) by the vertex-degree-sum theorem.

Specialization: A 2r-connected graph on \( n \) vertices at least \( rn \) edges.

Achieving this bound: \( H_{2r,n} \) is 2r-connected and has \( rn \) edges (\( 2r < n \)).

Construction: vertices are \([0],[1],...,[n-1]\) modulo \( n \).

Neighbors of the vertex \([a]\) are \([a+1]\) through \([a+r]\) and \([a-1]\) through \([a-r]\). These are distinct vertices: If \([a-p]=[a+q]\) then \( p+q \) is divisible by \( n \). But \( p+q \leq 2r < n \), hence \( p = q = 0 \). Contradiction. So the degree of \([a]\) is \( 2r \). This implies that \( H_{2r,n} \) has \( rn \) edges. Removing 2r vertices it is always possible to isolate a vertex and disconnect the graph. This makes \( \kappa_v(H_{2r,n}) \geq 2r \). Now we must show that if we remove \( 2r - 1 \) vertices the resulting graph is connected.

Remove \( 2r-1 \) vertices. Let \([u]\) and \([v]\) remain. If they are neighbors, we’re done. Otherwise, the circular path between them in either direction involves \( r \) or more internal vertices. Now in one of these directions, at least one internal vertex \([z]\) remains after cutting the \( 2r - 1 \) vertices. So can we can take an edge from \([u]\) to \([z]\). If the gap from \([z]\) to \([v]\) is still has \( r \) or more internal vertices, we can take an edge from \([z]\) to an internal vertex \([z']\) in the same direction. Keep on going, then eventually take an edge to \([v]\).

Section 5.3: Menger’s Theorems

\( u, v \) separating set \( S \): No path from \( u \) to \( v \) in \( G - S \). \( S \) could be set of vertices or set of edges.

Theorem 5.3.4 (Menger’s Theorem): Let \( u \neq v \) be such that \( uv \) is not an edge in a connected graph \( G \). Then the maximum number of internally disjoint \( uv \) paths in \( G \) is equal to the minimum number of vertices in a \( u, v \) separating set.
**Proof:** We’ll do the easy part now, and use Network Flows in Chapter 13 to do the hard part. We will just show that if \( P \) is a set of internally disjoint \( uv \) paths and \( S \) is any set of vertices whose removal separates \( u \) and \( v \), then \( |P| \leq |S| \).

Let \( p \) be one of the paths. At least one internal vertex of \( p \) must belong to \( S \), otherwise \( S \) doesn’t separate. So every \( p \in P \) contributes a vertex to \( S \), and since internally disjoint they must be distinct vertices. Hence \( |P| \leq |S| \).

Corollary: whenever \( |P| = |S| \) we must have \( |P| \) max and \( |S| \) min.

**Definition:** \( \kappa_v(s, t) \) = minimum number of vertices needed to separate \( s \) from \( t \), where \( s \neq t \) are not adjacent.

**Lemma 5.3.5:** \( \kappa_v(G) = \min \kappa_v(s, t) \).

**Proof:** Let \( \kappa_v(s, t) \) be minimum possible. If you can separate the graph with \( \kappa_v(s, t) - 1 \), then you can separate a pair of vertices with this many – contradiction. Therefore \( \kappa_v(G) \geq \kappa_v(s, t) \). On the other hand, removing \( \kappa_v(s, t) \) disconnects \( s \) from \( t \), hence disconnects the graph, hence \( \kappa_v(G) \leq \kappa_v(s, t) \). So they are equal.

**Theorem 5.3.6:** \( G \) is \( k \)-connected if and only if for each pair of vertices \( s \neq t \) there are at least \( k \) internally disjoint \( st \) paths in \( G \).

**Proof:** Assume \( G \) is \( k \)-connected. Let \( s, t \) be given, and suppose there are at most \( k - 1 \) internally disjoint \( st \) paths in \( G \). Since the maximum number of internally disjoint paths is \( < k \), by Menger’s Theorem we can say that the minimum number of vertices needed to separate \( s \) from \( t \) is \( < k \). This contradicts \( k \)-connected. Hence there at least \( k \) internally disjoint \( st \) paths in \( G \).

Conversely, suppose there are at least \( k \) internally disjoint \( st \) paths per \( s \neq t \) in \( G \). Suppose it is possible to separate \( G \) by removing \( k - 1 \) vertices. Then it is possible to separate a pair \( s, t \) by removing \( k - 1 \) vertices. But this leaves one of the paths intact – contradiction. Therefore \( G \) cannot be separated by removing \( k - 1 \) vertices, and \( G \) is \( k \)-connected.

**Corollary 5.3.7:** Let \( G \) be a \( k \)-connected graph and let \( v_0 \) through \( v_k \) be distinct vertices in \( G \). Then there are internally disjoint paths from \( v_0 \) to \( v_i \) for \( 1 \leq i \leq k \).

**Proof:** By the Expansion Lemma proved above, we can adjoin a new vertex \( V \) to \( G \) and edges from \( V \) to \( v_1 \) through \( v_k \), obtaining a \( k \)-connected graph \( G' \). There are at least \( k \) internally disjoint paths from \( v_0 \) to \( V \). This implies that there are exactly \( k \) internally disjoint paths of the type desired.

**Theorem 5.3.8:** Let \( G \) be \( k \) connected, where \( k \geq 3 \). Then any set of \( k \) vertices in \( G \) lives in a cycle of \( G \).

**Proof:** We will grow a cycle, using what we know about 2-connected graphs.

Let \( v_1 \) through \( v_k \) be any collection of \( k \) vertices. We will prove that there is a cycle through the first \( j \) of them by induction on \( j \geq 2 \).

\( j = 2 \): There is a cycle called \( C_2 \) through \( v_1 \) and \( v_2 \) since \( G \) is 2-connected.
Assume there is a cycle $C_j$ through $v_1$ through $v_j$, where $j < k$. If this cycle includes $v_{j+1}$ there is nothing to prove. If not, there are internally disjoint paths from $v_{j+1}$ through $v_1$ through $v_j$. So there are internally disjoint paths from $v_{j+1}$ to $v_1$ and $v_2$, and these don’t contain any other vertex on $C_j$. So we can build $C_{j+1}$ which incorporates $v_{j+1}$. See Figure 5.3.4, page 235.