Lectures Math 492 Spring 2010

Section 1.5: Number-Partitions

1.53: The Ferrers diagrams are composed of nested L-shapes, each of which has an odd number of dots.

1.54: Use Formula (1) with \( t = n - k \).

1.55: A bijection from \( P_d(n; k) \) to \( P(n - \binom{k}{2}; k) \) is

\[
\lambda_1 + \cdots + \lambda_k \mapsto (\lambda_1 - 0) + (\lambda_2 - 1) + \cdots + (\lambda_k - k + 1).
\]

1.56: Similar to the derivation of Formulas (1) and (2).

1.57: The Fibonacci numbers are \( f_1, f_2, f_3, \ldots \), where \( f_1 = f_2 = 1 \) and \( f_n = f_{n-1} + f_{n-2} \). They begin 1, 1, 2, 3, 5, 8, 13, .... For part (a), let \( a_n \) be the number of ordered partitions of \( n \) with parts of size \( \geq 2 \). Then \( a_2, a_3, a_4, a_5, \ldots = 1, 2, 3, 4, 5, \ldots. \) So a conjecture is \( a_n = f_{n-1} \). Prove this by induction on \( n \geq 2 \). For parts (b) and (c), do the same numerical comparison and prove your conjecture by induction.

1.58: Classify partitions of \( n \) according to those where the last part is 1 and those where the last part is \( \geq 2 \).

1.59: The solutions to \( x_1 + \cdots + x_k \leq n \) correspond to solutions to \( x_1 + \cdots + x_{k+1} = n \), i.e. to ordered partitions of \( n \) into \( k+1 \) parts. To count solutions where the \( x_i \)'s are non-negative, just add one to each part and \( k \) to \( n \).

1.60: This is equivalent to \( e(n) = o(n) + sc(n) = o(n) + odd/d\text{distinct}(n) \). We will construct an involution \( \phi : P(n) \to P(n) \) which fixes \( Odd/Distinct(n) \) and which maps \( E(n) \setminus Odd/Distinct(n) \) onto \( O(n) \). Given \( \lambda \in P(n) \), let \( e(\lambda) \) be its largest even part and let \( r(\lambda) \) be its largest repeated part (if either of these don’t exist, define as \(-\infty\) ). We will partition \( P(n) \) into \( P_>(n) \cup P_<(n) \cup P_=(n) \), where

\[
\begin{align*}
P_>(n) &= \{ \lambda \in P(n) : e(\lambda) > 2r(\lambda) \}, \\
P_<(n) &= \{ \lambda \in P(n) : e(\lambda) < 2r(\lambda) \}, \\
P_=(n) &= \{ \lambda \in P(n) : e(\lambda) = 2r(\lambda) \}.
\end{align*}
\]
Note that $P_-(n) = \text{Odd/Distinct}(n)$. We now define $\phi$ via

$$
\phi(\lambda) = \begin{cases} 
\lambda - e(\lambda) + \frac{e(\lambda)}{2} + \frac{e(\lambda)}{2} & \lambda \in P_>(n), \\
\lambda - r(\lambda) - r(\lambda) + 2r(\lambda) & \lambda \in P_<(n), \\
\lambda & \lambda \in P_=(n).
\end{cases}
$$

1.61: Let $n \geq 2$ and let $Q(n)$ be the set of partitions of $n$ into powers of 2. Then $P_=(n) = \emptyset$, and $\phi$ restricted to $P(n)$ establishes a 1:1 correspondence between $Q_e(n)$ and $Q_o(n)$. Therefore $|Q_e(n)| = |Q_o(n)|$.

1.63: Hook decomposition.