Method of Roots: Why it works

Let
\[ g(x) = \sum_{n=0}^{\infty} a_n x^n = \frac{R(x)}{1 + c_1 x + \cdots + c_k x^k} \]
be a generating function with the degree of \( R(x) \) strictly less than \( k \). Assume the polynomial
\[ x^k + c_1 x^{k-1} + \cdots + c_k = (x - r_1)(x - r_2) \cdots (x - r_k). \]

Then
\[ \frac{R(x)}{1 + c_1 x + \cdots + c_k x^k} = \frac{x^{-k} R(x)}{x^k + c_1 x^{-k+1} + \cdots + c_k} = \]
\[ \frac{x^{-k} R(x)}{(x^{-1} - r_1)(x^{-1} - r_2) \cdots (x^{-1} - r_k)} = \]
\[ \frac{R(x)}{x^k (1 - r_1 x)(1 - r_2 x) \cdots (1 - r_k x)}. \]

We can now obtain a formula for \( a_n \) after performing a partial fraction decomposition on the latter expression and extracting the coefficient of \( x^n \). If the factor \( 1 - rx \) appears to the \( j^{th} \) power in \( (1 - r_1 x)(1 - r_2 x) \cdots (1 - r_k x) \), then the root \( r \) contributes
\[ \frac{A_1}{1 - rx} + \frac{A_2}{(1 - rx)^2} + \cdots + \frac{A_j}{(1 - rx)^j} \]
to the partial fraction decomposition. The coefficient of \( x^n \) in this is
\[ A_1 r^n + A_2 \binom{n+1}{1} r^n + \cdots + A_j \binom{n+j-1}{j-1} r^n. \]

Multiplying out the binomial coefficients and combining like terms, this simplifies to
\[ B_1 r^n + B_2 n r^n + \cdots + B_j n^{j-1} r^n. \]

This expression will appear in the formula for \( a_n \), and there will be one such contribution for each distinct root of the polynomial \( x^k + c_1 x^{k-1} + \cdots + c_k \).
Conversely, suppose that
\[ f(n) = B_1 r^n + B_2 nr^n + \cdots + B_j n^{j-1} r^n. \]
Then after some algebra this can be re-written in the form
\[ f(n) = A_1 r^n + A_2 \binom{n + 1}{1} r^n + \cdots + A_j \binom{n + j - 1}{j - 1} r^n, \]
which is the coefficient of \( x^n \) in the generating function
\[ \frac{A_1}{1 - rx} + \frac{A_2}{(1 - rx)^2} + \cdots + \frac{A_j}{(1 - rx)^j}. \]
Therefore
\[ \sum_{n=0}^{\infty} f(n) x^n = \frac{A_1}{1 - rx} + \frac{A_2}{(1 - rx)^2} + \cdots + \frac{A_j}{(1 - rx)^j}. \]

Application: find a formula for \( a_n \) given that \( a_n - 3a_{n-1} = 2^n + n3^n, \)
\( a_0 = 1. \)

Solution: Let \( g(x) = \sum_{n=0}^{\infty} a_n x^n. \) By the analysis above, \( 2^n + n3^n \) is the coefficient of
\[ \frac{A}{1 - 2x} + \frac{B}{1 - 3x} + \frac{C}{(1 - 3x)^2} \]
for some choice of constants \( A, B, C. \) On the other hand,
\[ a_n - 3a_{n-1} \]
is the coefficient of \( x^n \) in \( g(x) - 3xg(x) + D \) for some choice of constant \( D. \)
Therefore
\[ g(x) - 3xg(x) + D = \frac{A}{1 - 2x} + \frac{B}{1 - 3x} + \frac{C}{(1 - 3x)^2}, \]
which implies
\[ g(x) = \frac{-D}{1 - 3x} + \frac{A}{(1 - 3x)(1 - 2x)} + \frac{B}{(1 - 3x)^2} + \frac{C}{(1 - 3x)^3}. \]
A partial fraction decomposition of this yields

\[ g(x) = \frac{\alpha}{1 - 3x} + \frac{\beta}{(1 - 3x)^2} + \frac{\gamma}{(1 - 3x)^3} + \frac{\delta}{1 - 2x}. \]

By the analysis above, this suggests that \( a_n \), the coefficient of \( x^n \) in \( g(x) \), has the form

\[ a_n = p3^n + qn3^n + rn^23^n + s2^n. \]

The constants \( p, q, r, s \) can be found using initial conditions. Where do these roots come from? One of root of 3 is supplied by the \( a_n - 3a_{n-1} \) side of the recurrence relation, and the other two roots of 3 and the root of 2 is supplied by the \( 2^n + n3^n \) side of the recurrence relation. In the method of roots, we circumvent the generating function and just concentrate on how the roots arise. (Note that some of the roots can be complex numbers.)