Extracting the coefficients of \( \frac{ax+b}{1+cx+dx^2} \)

Let \( g(x) = \sum_{n=0}^{\infty} a_n x^n \) be a generating function, and assume \( g(x) = \frac{ax+b}{1+cx+dx^2} \).

**Step 1:** Factor the denominator as \( 1 + cx + dx^2 = (1 - px)(1 - qx) \). To find \( p \) and \( q \), note that setting \( x = \frac{1}{p} \) yields

\[
1 + c \frac{1}{p} + d \frac{1}{p^2} = (1 - \frac{p}{p})(1 - \frac{q}{p}) = 0.
\]

Multiplying by \( p^2 \), obtain

\[
p^2 + cp + d = 0.
\]

The quadratic formula yields

\[
p = \frac{-c \pm \sqrt{c^2 - 4d}}{2}.
\]

Use one root for \( p \), the other root for \( q \).

**Step 2:** If \( p \neq q \), then partial fraction decomposition yields

\[
\frac{a + bx}{1 + cx + dx^2} = \frac{A}{1 - px} + \frac{B}{1 - qx}.
\]

\( A \) and \( B \) to be determined. Therefore

\[
[x^n] \frac{a + bx}{1 + cx + dx^2} = [x^n] \frac{A}{1 - px} + [x^n] \frac{B}{1 - qx},
\]

\[
a_n = A \cdot p^n + B \cdot q^n.
\]

Now \( A \) and \( B \) can be determined by plugging in the first two values of \( n \) into the formula for \( a_n \).

**Step 3:** If \( p = q \), then partial fraction decomposition yields

\[
\frac{a + bx}{1 + cx + dx^2} = \frac{A}{1 - px} + \frac{B}{(1 - px)^2}.
\]

\( A \) and \( B \) to be determined. Therefore

\[
[x^n] \frac{a + bx}{1 + cx + dx^2} = [x^n] \frac{A}{1 - px} + [x^n] \frac{B}{(1 - px)^2},
\]

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\[ a_n = A \cdot p^n + B \cdot \left( \binom{n+1}{1} \right) p^n = (A + B(n + 1))p^n. \]

Now \( A \) and \( B \) can be determined by plugging in the first two values of \( n \) into the formula for \( a_n \).

**Example 1:** \( a_0 = 1, \ a_1 = 3, \ a_n = 4a_{n-1} + a_{n-2} \). The generating function is

\[ g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \\
1 + 3x + (4a_1 + 5a_0) x^2 + (4a_2 + 5a_1) x^3 + \cdots = \\
1 + 3x + 4x(a_1 x + a_2 x^2 + \cdots) + 5x^2(a_0 + a_1 x + a_2 x^2 + \cdots = \\
1 + 3x + 4x(g(x) - 1) + 5x^2 g(x). \]

Solving for \( g(x) \) we obtain

\[ g(x) = \frac{1 + 2x}{1 - 4x - 5x^2}. \]

Solving \( p^2 - 4p - 5 \) we obtain \( p = 5 \) and \( q = -1 \). Therefore

\[ a_n = A \cdot 5^n + B \cdot (-1)^n. \]

Plugging in \( n = 0 \) and \( n = 1 \) yields

\[ 1 = A + B, \]
\[ 3 = 5A - B. \]

Therefore \( A = \frac{2}{3} \) and \( B = \frac{1}{3} \) and

\[ a_n = \frac{2 \cdot 5^n + (-1)^n}{3}. \]

**Example 2:** \( a_0 = 1, \ a_1 = 3, \ a_n = 4a_{n-1} - 4a_{n-2} \). The generating function is

\[ g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \\
1 + 3x + (4a_1 - 4a_0) x^2 + (4a_2 + 4a_1) x^3 + \cdots = \\
1 + 3x + 4x(a_1 x + a_2 x^2 + \cdots) - 4x^2(a_0 + a_1 x + a_2 x^2 + \cdots = \\
1 + 3x + 4x(g(x) - 1) - 4x^2 g(x). \]
Solving for $g(x)$ we obtain

$$g(x) = \frac{1 - x}{1 - 4x + 4x^2}.$$

Solving $p^2 - 4p + 4$ we obtain $p = 2$ and $q = 2$. Therefore

$$a_n = (A + B(n + 1))2^n$$

Plugging in $n = 0$ and $n = 1$ yields

$$1 = A + B,$$

$$3 = 2A + 4B.$$ 

Therefore $A = \frac{1}{2}$ and $B = \frac{1}{2}$ and

$$a_n = (n + 2)2^{n-1}.$$