Expected number of leading 1s in binary strings of length $n$

There are $2^n$ binary strings of length $n$. There are $2^{n-1}$ of these with no leading 1s, $2^{n-2}$ of these with 1 leading 1, $2^{n-3}$ of these with 2 leading 1s, ..., $2^{n-n}$ of these with $n-1$ leading 1s, and 1 of these with $n$ leading 1s. The average of these numbers is

$$\frac{2^{n-2} \cdot 1 + 2^{n-3} \cdot 2 + \cdots + 2^{n-n} \cdot (n-1) + n}{2^n}.$$

We wish to simplify the expression

$$2^{n-2} \cdot 1 + 2^{n-3} \cdot 2 + \cdots + 2^{n-n} \cdot (n-1).$$

We can group this into

$$1 + (1 + 2) + (1 + 2 + 2^2) + \cdots + (1 + 2 + \cdots + 2^{n-2}) =$$

$$(2^1 - 1) + (2^2 - 1) + \cdots + (2^{n-1} - 1) =$$

$$(2^1 + 2^2 + \cdots + 2^{n-1}) - (n - 1) =$$

$$2(1 + 2 + \cdots + 2^{n-2}) - n + 1 =$$

$$2(2^{n-1} - 1) - n + 1 =$$

$$2^n - n - 1.$$  

Therefore the average number of leading 1s is

$$\frac{2^n - n - 1 + n}{2^n} = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}.$$  

For example, when $n = 3$ we have an average of $1 - \frac{1}{8} = \frac{7}{8} = 0.875$ leading 1s per binary string of length 3.