CompleteGraph[n_] :=
Module[{vertexSet, graph, i, vertex, neighborhood, graphEntry},
  vertexSet = Table[i, {i, 1, n}];
  graph = {};
  For[i = 1, i <= Length[vertexSet], i++,
    vertex = i;
    neighborhood = Complement[vertexSet, {i}];
    graphEntry = {vertex, neighborhood};
    graph = Append[graph, graphEntry];
  ];
  graph
]
CompleteGraph[5]

(PathGraph[n_] :=
Module[{graph, vertex, neighborhood, graphEntry},
  graph = {};
  vertex = 1;
  neighborhood = {2};
  graphEntry = {vertex, neighborhood};
  graph = Append[graph, graphEntry];
  vertex = n;
  neighborhood = {n - 1};
  graphEntry = {vertex, neighborhood};
  graph = Append[graph, graphEntry];
  For[vertex = 2, vertex <= n - 1, vertex++,
    neighborhood = {vertex - 1, vertex + 1};
    graphEntry = {vertex, neighborhood};
    graph = Append[graph, graphEntry];
  ];
  graph
]
PathGraph[5]

\{\{1, \{2\}\}, \{5, \{4\}\}, \{2, \{1, 3\}\}, \{3, \{2, 4\}\}, \{4, \{3, 5\}\}\}\\

CycleGraph[n_] :=
Module[{graph, vertex, neighborhood, graphEntry},
  graph = {};
  vertex = 1;
  neighborhood = \{2, n\};
  graphEntry = \{vertex, neighborhood\};
  graph = Append[graph, graphEntry];
  vertex = n;
  neighborhood = \{n - 1, 1\};
  graphEntry = \{vertex, neighborhood\};
  graph = Append[graph, graphEntry];
  For[vertex = 2, vertex <= n - 1, vertex++,
    neighborhood = \{vertex - 1, vertex + 1\};
    graphEntry = \{vertex, neighborhood\};
    graph = Append[graph, graphEntry];
  ];
  graph
]

CycleGraph[5]

\{\{1, \{2, 5\}\}, \{5, \{4, 1\}\}, \{2, \{1, 3\}\}, \{3, \{2, 4\}\}, \{4, \{3, 5\}\}\}\\

WheelGraph[n_] :=
Module[{graph, i, vertex, neighborhood, graphEntry},
  graph = CycleGraph[n - 1];
  For[i = 1, i <= n - 1, i++,
    graph[[i]][[2]] = Append[graph[[i]][[2]], n];
  ];
  vertex = n;
  neighborhood = Table[i, \{i, 1, n - 1\}];
  graphEntry = \{vertex, neighborhood\};
  graph = Append[graph, graphEntry];
  graph
]
WheelGraph[5]
\[\\{1, \{2, 4, 5\}\}, \{4, \{3, 1, 5\}\}, \{2, \{1, 3, 5\}\}, \{3, \{2, 4, 5\}\}, \{5, \{1, 2, 3, 4\}\}\]

FindEdge[graph_] := 
Module[{edgeFound, output, index, graphEntry, vertex, neighborhood, neighbor},
    edgeFound = False;
    output = {};
    index = 1;
    While[index \leq \text{Length}[graph] \&\& \text{edgeFound} \!\= \text{False},
        graphEntry = graph[[index]];
        vertex = graphEntry[[1]];
        neighborhood = graphEntry[[2]];
        If[\text{Length}[neighborhood] > 0,
            edgeFound = True;
            neighbor = neighborhood[[1]];
        ];
        index ++;
    ];
    If[edgeFound \!\= \text{True},
        output = Append[output, \{vertex, neighbor\}];
    ];
    output
]

FindEdge[WheelGraph[5]]
\[\\{1, 2\}\]

DeleteEdge[graph_, edge_] := 
Module[{x, y, newGraph, i, graphEntry, vertex, neighborhood, newNeighborhood},
    x = edge[[1]];
    y = edge[[2]];
    newGraph = {};
    For[i = 1, i \leq \text{Length}[graph], i ++,
        graphEntry = graph[[i]];
        vertex = graphEntry[[1]];
        neighborhood = graphEntry[[2]];
        Switch[\{vertex \!\= x, vertex \!\= y\},
            \{True, False\},
            newNeighborhood = \text{Complement}[neighborhood, \{y\}],
            \{False, True\},
            newNeighborhood = \text{Complement}[neighborhood, \{x\}],
            \{False, False\},
            newNeighborhood = neighborhood
        ];
        newGraph = Append[newGraph, \{vertex, newNeighborhood\}];
    ];
    newGraph
]
DeleteEdge[WheelGraph[5], {1, 2}]

\{\{1, \{4, 5\}\}, \{4, \{3, 1, 5\}\}, \{2, \{3, 5\}\}, \{3, \{2, 4, 5\}\}, \{5, \{1, 2, 3, 4\}\}\}

IdentifyVertices[graph_, edge_] :=
Module[
{x, y, xyNeighborhood, i, graphEntry, vertex, neighborhood, newGraph, newNeighborhood},
x = edge[[1]]; y = edge[[2]]; xyNeighborhood = {};
For[i = 1, i \leq \text{Length}[graph], i ++,
  graphEntry = graph[[i]]; vertex = graphEntry[[1]]; neighborhood = graphEntry[[2]];
  Switch[\{vertex \begin{array}{l} x \\ y \end{array}, \begin{array}{l} x \\ y \end{array}<\text{True}, \text{False}\},
    \{\text{True, False}\},
    xyNeighborhood = \text{Union}[xyNeighborhood, \text{Complement}[neighborhood, \{y\}]],
    \{\text{False, True}\},
    xyNeighborhood = \text{Union}[xyNeighborhood, \text{Complement}[neighborhood, \{x\}]]\};
]
newGraph = {};
For[i = 1, i \leq \text{Length}[graph], i ++,
  graphEntry = graph[[i]]; vertex = graphEntry[[1]]; neighborhood = graphEntry[[2]];
  Switch[\{vertex \begin{array}{l} x \\ y \end{array}, \begin{array}{l} x \\ y \end{array}<\text{True}, \text{False}\},
    \{\text{True, False}\},
    newGraph = Append[newGraph, \{x, xyNeighborhood\}],
    \{\text{False, True}\},
    newNeighborhood = \text{Complement}[neighborhood, \{y\}];
  If[\text{Length}[newNeighborhood] < \text{Length}[neighborhood],
    newNeighborhood = \text{Union}[newNeighborhood, \{x\}]\};
  newGraph = Append[newGraph, \{vertex, newNeighborhood\}];
]
newGraph]
IdentifyVertices[WheelGraph[5], {1, 5}]
{{1, {2, 3, 4}, {4, {1, 3}}, {2, {1, 3}}, {3, {1, 2, 4}}}

ChromaticPolynomial[graph_, k_] :=
Module[{edgeList, polynomial, edge, H},
  edgeList = FindEdge[graph];
  Switch[Length[edgeList],
    0, polynomial = k^Length[graph],
    1, edge = edgeList[[1]];
    H[1] = DeleteEdge[graph, edge];
    H[2] = IdentifyVertices[graph, edge];
    polynomial = ChromaticPolynomial[H[1], k] - ChromaticPolynomial[H[2], k];
  ];
  polynomial
]

ChromaticPolynomial[CompleteGraph[5], k]
24 k - 50 k^2 + 35 k^3 - 10 k^4 + k^5

Factor[%]
(-4 + k) (-3 + k) (-2 + k) (-1 + k) k

Solve[% = 0, k]
{{k -> 0}, {k -> 1}, {k -> 2}, {k -> 3}, {k -> 4}}

ChromaticPolynomial[PathGraph[5], k]
k - 4 k^2 + 6 k^3 - 4 k^4 + k^5

Factor[%]
(-1 + k)^4 k

Solve[% = 0, k]
{{k -> 0}, {k -> 1}, {k -> 1}, {k -> 1}, {k -> 1}}

ChromaticPolynomial[CycleGraph[5], k]
4 k - 10 k^2 + 10 k^3 - 5 k^4 + k^5

Factor[%]
(-2 + k) (-1 + k) k (2 - 2 k + k^2)

Solve[% = 0, k]
{{k -> 0}, {k -> 1}, {k -> 1}, {k -> 1}, {k -> 1}}

ChromaticPolynomial[WheelGraph[10], k]
-510 k + 2303 k^2 - 4608 k^3 + 5376 k^4 - 4032 k^5 + 2016 k^6 - 672 k^7 + 144 k^8 - 18 k^9 + k^10
Factor[%]  
\((-2 + k) \cdot (-1 + k) \cdot k \left(7 - 5k + k^2\right)\)

Solve[\% = 0, k]  
\(\{k \rightarrow 0\}, \{k \rightarrow 1\}, \{k \rightarrow 2\}, \{k \rightarrow 2 - 1\}, \{k \rightarrow 2 + 1\}, \{k \rightarrow 3\},\)  
\(\{k \rightarrow 2 - (-1)^{3/4}\}, \{k \rightarrow 2 + (-1)^{3/4}\}, \{k \rightarrow 2 - (-1)^{3/4}\}, \{k \rightarrow 2 + (-1)^{3/4}\}\)