Section 1.1: Graph Models

Graph: \( G = (V, E) \)

Vertices, edges, adjacent vertices

Directed graph

Path: sequence of distinct vertices joined by edges

Circuit: path followed by starting vertex connected by edge

Connected graph: there is a path connecting every pair of vertices

Graph models: bipartite graph, vertex connectivity, edge connectivity, tree, edge cover, independent set of vertices, chromatic number, interval graph, vertex basis for digraph.

Matching example 1, Figure 1.2, page 5: are there 5 vertex-disjoint edges?

Dictionary search example 2, Figure 1.3, page 5: Look up definition of a word in a 1000 page dictionary. Turning the pages in order, you may have to examine 1000 pages, for an average of 500 pages per lookup. But if you open the book to page 500, you can tell the word is on that page, on an earlier page, or on a later page. One could conceivably have to check pages 500, 250, 125, 62, 31, 15, 7, 3, 1, for no greater than 9 pages to look at, or on average 4.5. More generally, \( n \) pages, \( \log_2 n \) in the worst case, half this on average.

Network reliability example 3, Figure 1.4, page 6: edges represent telephone lines, vertices represent switching stations. To what degree is this network connected? Vertex connectivity: what is the fewest number of vertices whose removal creates disconnected graph? 2 (\( e \) and \( k \)). Edge connectivity: what is the fewest number of edges whose removal creates disconnected graph? 2 (edges incident to \( a \)). Measure connectivity of a network this way. Edge connectivity 1: graph is a tree and \( e = v - 1 \).

Street surveillance example 4, Figure 1.4: Station police at street corners (vertices) to observe all streets (adjacent edges). Using all 6 vertices of degree 3 works, but we can omit vertex \( f \). 5 is minimal number of vertices since 4 vertices anywhere see at most 12 edges.
edge cover: set of vertices such that every edge has endpoint in it.

Scheduling meetings example 5, Figure 1.4: vertex = committee, edge = committees share member hence cannot meet at the same time. Any collection of committees with no edges in common can meet at the same time. Color these 1. Next independent collection of committees: color these 2. Keep on going. More generally, assign colors to graph in such a way that no two vertices assigned same color have edge between them. Same color vertices are independent. What is minimum number of colors needed?

independent set of vertices: no edges connecting them.

Relationship between edge covers and independent sets: If \(C\) is a cover then \(V - C\) is an independent set, and if \(I\) is an independent set then \(V - I\) is a cover. Reason: Suppose \(a, b \in V - C\). If edge between them, one or the other endpoint must be in \(C\) – contradiction. Hence \(V - C\) is independent. Now let \(\{a, b\}\) be an edge. Then \(a\) or \(b\) cannot be in \(I\), therefore one endpoint of the edge is in \(V - I\), hence \(V - I\) is a cover.

Corollary: if \(C\) is minimal edge cover then \(V - C\) is maximal independent set.

Interval graph modeling example 6, page 9: Consider species coexisting in a tidepool. Vertex = species, edge = species in competition for food. Now see if it’s possible to express graph as interval graph. If it’s possible in principle, one might try: height above bottom, or average immersion, or average exposure to sun. These are numerical parameters that have a natural range of values per species. But if it’s not possible in principle, don’t bother.

Influence model example 7, page 10: vertex = person, directed edge = influence. Minimal set of people influencing the group: vertices which have directed paths to all vertices. Called vertex basis. To aid in finding, create digraph whose edges represent paths in original graph. Now find minimal vertex cover of vertices.

Section 1.2: Isomorphism

New graph from old: Let \(G = (V, E)\) be a graph. Let \(\phi : V \to V'\) be an injective function. Then we get a new graph \(G' = (V', E')\) in the natural way. The two graphs \(G\) and \(G'\) are isomorphic. Note that they have the same shape.
Two graphs $G$ and $H$ are isomorphic if the function $\phi$ can be found. $G$ and $H$ don’t necessarily have the same shape to begin with: two ways to draw $K_4$. But if $\phi$ can be found, they really can be drawn to look the same.

How to find: try redrawing $H$ to look like $G$. Take into consideration properties they have in common: number of vertices, number of edges, degree sequence, longest paths, number of cycles, longest cycle, sizes of cycles, other subgraphs, etc.

How to prove that two graphs are not isomorphic: one could attempt to redraw $H$ to look like $G$ and fail every time. But this is not a definitive proof because there may be something you’ve overlooked. Instead, show that $H$ has a feature not in $G$. If there really were an isomorphism, this feature would be preserved – contradiction.

Why isomorphism matters: you wouldn’t want to patent a design that has already been discovered. Or: you want to prove something about all graphs of a certain size, and you wouldn’t want to waste time on isomorphic copies.

Example 1, page 18, figure 1.8: not isomorphic.

Example 2, page 18, figure 1.9: try looking at the neighbors of $a$ and the neighbors of 1. Now look at the triangles containing $a$ and 1. $abc, afg, abg, 145, 147, 125$. Suggests $b \leftrightarrow 4$ and $f \leftrightarrow 2$. Keep on going. Another approach: inspect graph complements. Prove the statement rigorously.

Isomorphism of directed graphs: same idea.

Example 3, page 20, figure 1.12: left graph has directed 8-cycle, right graph doesn’t.