1. What is the probability that a random rearrangement of the word MATHEMATICS has all of its vowels side-by-side and at least one letter separating the Ms?

**Solution:** The vowels are AAEI and the consonants are MMTTHCS. We will find the number of rearrangements of VMMMTTHCS where V stands for the block of vowels and the Ms are separated, then replace the symbol V by an arrangement of the vowels.

Stage 1: Count the number of spacings of the Ms among the symbols VMMMTTHCS. Solutions to \( x_1 + x_2 + x_3 = 6 \) with \( x_1 \geq 0, x_2 \geq 1, x_3 \geq 0 \). Same as solutions to \( (x_1) + (x_2 - 1) + (x_3) = 5 \), each number in parentheses \( \geq 0 \). Same as rearrangements of 5 1s and 2 +s, or \( \frac{7!}{5!2!} \).

Stage 2: Count the number of ways to rearrange VTHCS in the available slots: \( \frac{6!}{1!1!1!1!1!} \).

Stage 3: Count the number of ways to replace V by an arrangement of AAEI: \( \frac{4!}{2!1!1!} \).

Total number of restricted rearrangements: \( \frac{7!}{5!2!} \cdot \frac{6!}{1!1!1!1!1!} \cdot \frac{4!}{2!1!1!} \).

Total number of unrestricted rearrangements of MATHEMATICS: \( \frac{11!}{2!1!1!1!2!2!1!1!1!} \).

Probability of restricted rearrangements: after dividing and canceling,

\[
\frac{7!6!4!}{5!11!} = \frac{1}{55}.
\]
2. Using the Binomial Theorem, simplify $\sum_{k=0}^{n} k^2 \binom{n}{k} 3^k$.

Solution:

\[(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\]

\[(x + 1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k\]

\[n(x + 1)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k x^{k-1}\]

\[nx(x + 1)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k x^k\]

\[n(x + 1)^{n-1} + n(n-1)x(x + 1)^{n-2} = \sum_{k=0}^{n} \binom{n}{k} k^2 x^{k-1}\]

\[nx(x + 1)^{n-1} + n(n-1)x^2(x + 1)^{n-2} = \sum_{k=0}^{n} \binom{n}{k} k^2 x^k\]

\[n34^{n-1} + n(n-1)3^2 4^{n-2} = \sum_{k=0}^{n} \binom{n}{k} k^2 3^k\]

\[12n4^{n-2} + (9n^2 - 9n)4^{n-2} = \sum_{k=0}^{n} \binom{n}{k} k^2 3^k\]

\[(9n^2 + 3n)4^{n-2} = \sum_{k=0}^{n} \binom{n}{k} k^2 3^k\]
3. (a) Consider all subsets of size $k$ chosen from $\{1, 2, \ldots, n\}$. These can be organized as follows: some of them contain both 1 and 2, some of them contain 1 but not 2, some of them contain 2 but not 1, and some of them contain neither 1 nor 2. Explain how you can use this observation to give a combinatorial proof of the identity

$$\binom{n}{k} = \binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k}.$$ 

(b) Adapt this argument to prove

$$\binom{n}{k} = \sum_{i=0}^{p} \binom{p}{i} \binom{n-p}{k-i}.$$ 

Solution: There are $\binom{n}{k}$ subsets of size $k$. We can count these by cases according to how many of the numbers in the range $\{1, \ldots, p\}$ fall into a subset. In case $i$, there are $\binom{p}{i}$ ways to select $i$ of these numbers followed by $\binom{n-p}{k-i}$ ways to select $k - i$ numbers from the range $\{p + 1, \ldots, n\}$, for a total of $\binom{p}{i} \binom{n-p}{k-i}$. Summing over all possible cases, $0 \leq i \leq p$, we get the sum formula for $\binom{n}{k}$. For part (a) we are using $p = 2$. 

3
4. (a) Find the generating function model for the number of solutions to
\[ e_1 + e_2 + e_3 + e_4 = r, \]
where each of the variables is an integer in the range \( \{0, 1, 2, \ldots, 30\} \).

(b) Now extract coefficient of \( x^{100} \) to find the number of solutions when \( r = 100 \).

**Solution:** The generating function model is
\[
(x^0 + x^1 + \cdots + x^{30})^4 = \left(\frac{1 - x^{31}}{1 - x}\right)^4 = \frac{1 - 4x^{31} + 6x^{62} - 4x^{93} + x^{124}}{(1 - x)^4}.
\]

To compute the coefficient of \( x^{100} \), we have
\[
\frac{1 - 4x^{31} + 6x^{62} - 4x^{93} + x^{124}}{(1 - x)^4} = (1-4x^{31}+6x^{62}-4x^{93}+x^{124})\sum_{n=0}^{\infty} \binom{n+3}{3} x^n =
\sum_{n=0}^{\infty} \binom{n+3}{3} x^n - 4 \sum_{n=0}^{\infty} \binom{n+3}{3} x^{n+31} + 6 \sum_{n=0}^{\infty} \binom{n+3}{3} x^{n+62} - 4 \sum_{n=0}^{\infty} \binom{n+3}{3} x^{n+93} + \sum_{n=0}^{\infty} \binom{n+3}{3} x^{n+124}.
\]

Therefore
\[
[x^{100}] = \binom{100+3}{3} - 4 \binom{69+3}{3} + 6 \binom{38+3}{3} - 4 \binom{7+3}{3} =
\binom{103}{3} - 4 \binom{72}{3} + 6 \binom{41}{3} - 4 \binom{10}{3}.
\]
5. Find the number of ways to put 100 different people into 6 different buses, an odd number in each bus. The order of the people in each bus doesn’t matter, but the identities of the people in each bus does.

Solution: The number of ways to place $e_i$ people in bus $i$, $1 \leq i \leq 6$, is equal to the number of rearrangements $e_1 1s$, $e_2 2s$, $e_3 3s$, $e_4 4s$, $e_5 5s$, and $e_6 6s$, namely $\frac{100!}{e_1!e_2!e_3!e_4!e_5!e_6!}$. Therefore the total number of ways to load the bus, summing over all the cases, is

$$\sum_{e_1, e_2, e_3, e_4, e_5, e_6 \in \{1, 3, 5, \ldots\}} \frac{100!}{e_1!e_2!e_3!e_4!e_5!e_6!}.$$

This equal to the coefficient of $x^{100}$ in

$$100! \left( \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \right)^6 = 100! \left( \frac{e^x - e^{-x}}{2} \right)^6 =$$

$$\frac{100!}{26} \sum_{k=0}^{6} \binom{6}{k} (e^x)^k (-e^{-x})^{6-k} = \frac{100!}{26} \sum_{k=0}^{6} \binom{6}{k} (-1)^{6-k} e^{(2k-6)x},$$

which is equal to

$$[x^{100}] = \frac{100!}{26} \sum_{k=0}^{6} \binom{6}{k} \frac{(-1)^{6-k}(2k-6)^{100}}{100!} = \frac{1}{26} \sum_{k=0}^{6} \binom{6}{k} (-1)^{6-k}(2k-6)^{100}$$

$$= \frac{6^{100} - 6 \cdot 4^{100} + 15 \cdot 2^{100} - 20 \cdot 0^{100} + 15 \cdot 2^{100} - 6 \cdot 4^{100} + 6^{100}}{26} = \frac{6^{100} - 6 \cdot 4^{100} + 15 \cdot 2^{100}}{25}.$$