39. (a) Let $y$ represent percentage of population which has heard rumor at time $t$ on a 0-100 scale. Let $t$ measure number of hours since 8:00 am. Then the differential equation is

$$\frac{dy}{dt} = k \frac{y}{100} \frac{100 - y}{100} = 0.0001ky(100 - y).$$

Separating variables,

$$\frac{dy}{y(100 - y)} = 0.0001k \, dt.$$

(b) Partial fraction decomposition:

$$\frac{1}{y(100 - y)} = \frac{p}{y} + \frac{q}{100 - y}$$

$$1 = p(100 - y) + qy = 100p + (q - p)y$$

$$100p = 1, \quad q - p = 0$$

$$p = \frac{1}{100}, \quad q = \frac{1}{100}$$

$$\int \frac{1}{100} \frac{1}{y} + \frac{1}{100} \frac{1}{100 - y} \, dy = \int 0.0001k \, dt$$

$$\frac{1}{100} \ln y - \frac{1}{100} \ln 100 - y = .0001kt + C$$

$$\ln \frac{y}{100 - y} = .01kt + 100C$$

When $t = 0$, $y = 8$. This yields

$$\ln \frac{8}{92} = 100C, \quad \ln \frac{y}{100 - y} = .01kt + \ln \frac{8}{92}.$$ 

When $t = 4$, $y = 50$. This yields

$$\ln \frac{50}{50} = .04k + \ln \frac{8}{92}, \quad k = -25 \ln \frac{8}{92}, \quad \ln \frac{y}{100 - y} = -0.25 \ln \frac{8}{92}t + \ln \frac{8}{92}.$$
So now we have

\[
\ln \frac{y}{100 - y} = (1 - 0.25t) \ln \frac{8}{92} = \ln \left( \frac{8}{92} \right)^{1-0.25t}
\]

\[
\frac{y}{100 - y} = \left( \frac{8}{92} \right)^{1-0.25t}
\]

\[
y = 100 \left( \frac{8}{92} \right)^{1-0.25t} \left( 1 + \left( \frac{8}{92} \right)^{1-0.25t} \right).
\]

(c) We will use the equation

\[
\ln \frac{y}{100 - y} = (1 - 0.25t) \ln \frac{8}{92}
\]

from an earlier calculation. When \( y = 90 \), we have

\[
\ln \frac{90}{10} = (1 - 0.25t) \ln \frac{8}{92}
\]

which yields

\[
t = 7.59855.
\]

Therefore the rumor spreads to 90 percent of the population roughly 7.6 hours after 8:00 am, which would make it 3:36 pm.

45. Let \( y \) represent gallons of alcohol in the tank at time \( t \) minutes. Then

\[
\frac{dy}{dt} = \text{rate in} - \text{rate out}
\]

in gallons of alcohol \( \text{min} \). Beer is flowing in at a fluid rate of \( 5 \) gallons of beer \( \text{min} \) with a concentration of \( 0.06 \) gallons of alcohol gallons of beer, the product of which yields an alcohol rate of \( 0.3 \) gallons of alcohol \( \text{min} \). Beer is flowing out at a fluid rate of \( 5 \) gallons of beer \( \text{min} \) with a concentration of \( y \) gallons of alcohol gallons of beer, the product of which yields an alcohol rate of \( 0.01y \) gallons of alcohol \( \text{min} \). Therefore

\[
\frac{dy}{dt} = 0.3 - 0.01y.
\]
Since the initial alcohol content in the tank is 4% and we are pouring in 6% beer, the alcohol content in the tank continues to increase, which implies that the quantity \( .3y - .01y \) will always be positive. Note also that at time \( t = 0 \) we have \( y = (.04)(500) = 20 \) gallons of alcohol.

Solve the differential equation as follows:

\[
\frac{dy}{0.3 - 0.01y} = dt
\]

\[
\int \frac{dy}{0.3 - 0.01y} = \int dt
\]

\[
-100 \int \frac{du}{u} = \int dt \quad u = 0.3 - 0.01y
\]

\[
-100 \ln |u| = t + C
\]

\[
-100 \ln(0.3 - 0.01y) = t + C.
\]

Using \( t = 0, \ y = 20 \), we obtain \( C = 230 \) (rounding off). Therefore

\[
-100 \ln |0.3 - 0.01y| = t + 230.259.
\]

After 1 hour, \( t = 60 \), which implies

\[
-100 \ln(0.3 - 0.01y) = 290.259
\]

\[
\ln(.3-.01y) = -2.90259
\]

\[
(.3-.01y) = e^{-2.90259} = 0.0548809,
\]

hence \( y = 24.5119 \) gallons of alcohol after 1 hour. The alcohol content at this time is \( 100 \times \frac{24.5}{500} = 4.09238 \) percent by volume.