Show all work

1. (a) Sketch the curve $x(t) = t^3 + t^2$, $y(t) = t^3 - t^2 + 1$, $0 \leq t \leq 3$.

(b) Find the equation of the line tangent to the curve at time $t = 2$.

Solution: The curve passes through the point $(12, 5)$ at time $t = 2$. At this time, the slope of the tangent line is

$$\frac{y'(t)}{x'(t)} = \frac{3t^2 - 2t}{3t^2 + 2t} = \frac{1}{2}.$$

So the equation of the tangent line is

$$y - 5 = \frac{1}{2}(x - 12)$$

or

$$y = \frac{1}{2}x - 1.$$
2. Find the length of the curve \( x(t) = t^3 + t^2, \ y(t) = t^3 - t^2 + 1, \ 0 \leq t \leq 3. \)

To evaluate the integral, simplify and factor the expression under the square root symbol, then use an appropriate \( u \)-substitution.

**Solution:** The length of the curve is

\[
L = \int_0^3 \sqrt{(3t^2 + 2t)^2 + (3t^2 - 2t)^2} \, dt = \int_0^3 \sqrt{18t^4 + 8t^2} \, dt = \int_0^3 t\sqrt{18t^2 + 8} \, dt.
\]

Now set \( u = 18t^2 + 8, \ du = 36t \, dt. \) Then

\[
L = \frac{1}{36} \int_8^{170} u^{\frac{1}{2}} \, du = \left. \frac{1}{54} u^{\frac{3}{2}} \right|_8^{170} = \frac{1}{54} \left( 170^3 - 8^3 \right).
\]
3. Find the area underneath the curve $x(t) = t^3 + t^2$, $y(t) = t^3 - t^2 + 1$, $0 \leq t \leq 3$ and above the $x$-axis.

Solution:

\[ A = \int_0^3 y(t)x'(t) \, dt = \int_0^3 (t^3 - t^2 + 1)(3t^2 + 2t) \, dt = \]

\[ = \int_0^3 2t + 3t^2 - 2t^3 - t^4 + 3t^5 \, dt = \left. \frac{2t^2}{2} + \frac{3t^3}{3} - \frac{2t^4}{4} - \frac{t^5}{5} + \frac{3t^6}{6} \right|_0^3 = \frac{1557}{5} = 311.4 \]
4. (a) Sketch the two polar curves \( r = \theta \) and \( r = \frac{\pi}{2} - \theta \) in the same coordinate system, \( 0 \leq \theta \leq \frac{\pi}{2} \).

(b) Find the polar coordinates of the two points of intersection of these two curves in the first quadrant.

(c) Calculate the area trapped between the two curves in the first quadrant.

**Solution:** The two curves are spirals which intersect at the origin and at \( \left( \frac{\pi}{4}, \frac{\pi}{4} \right) \) in the first quadrant. Using symmetry, the area is

\[
A = 2 \int_0^{\frac{\pi}{2}} \frac{\theta^2}{2} d\theta = 2 \left. \frac{\theta^3}{6} \right|_0^{\frac{\pi}{4}} = \frac{\pi^3}{192}.
\]
5. Find the center and radius of the sphere with equation

\[ 2x^2 + 2y^2 + 2z^2 = 4x - 8y + 16z. \]

**Solution:** The equation in standard form is

\[ 2x^2 + 2y^2 + 2z^2 - 4x + 8y - 16z = 0, \]
\[ x^2 + y^2 + z^2 - 2x + 4y - 8z = 0, \]
\[ (x - 1)^2 + (y + 2)^2 + (z - 4)^2 = 21. \]

Therefore \( C = (1, -2, 4), \) \( R = \sqrt{21}. \)
6. Find the angle in degrees between the vectors $u = \langle 2, 3, 4 \rangle$ and $v = \langle 3, 4, 5 \rangle$.

**Solution:**

$$\theta = \cos^{-1} \frac{6 + 12 + 20}{\sqrt{4 + 9 + 16} \sqrt{9 + 16 + 25}} = \cos^{-1} \frac{38}{\sqrt{29} \sqrt{50}} = 3.69^\circ$$
7. Find the parametric and symmetric equations for the line that passes through the points \( P(4, 3, 2) \) and \( Q(3, 4, 5) \).

Solution:

\[
x(t) = 4 - t, \quad y(t) = 3 + t, \quad z(t) = 2 + 3t
\]

\[
4 - x = y - 3 = \frac{z - 2}{3}.
\]
8. Using the cross-product method, find the equation of the plane that passes through the points \( P(2, 3, 4), Q(4, 3, 2), \) and \( R(3, 4, 5) \).

\[
\begin{align*}
\mathbf{u} &= \mathbf{Q} - \mathbf{P} = (2, 0, -2), \quad \mathbf{v} = \mathbf{R} - \mathbf{P} = (1, 1, 1) \\
\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} i & j & k \\ 2 & 0 & -2 \\ 1 & 1 & 1 \end{vmatrix} = 2i - 4j + 2k = (2, -4, 2) \\
2(x - 2) - 4(y - 3) + 2(z - 4) &= 0 \\
2x - 4y + 2z &= 0 \\
x - 2y + z &= 0
\end{align*}
\]