1. Using the Alternating Series Estimation Theorem, approximate the value of the infinite series with $|\text{error}| < 0.001$:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^5}.$$ 

**Solution:** According to the Alternating Series Estimation Theorem, the $n^{th}$ partial sum $s_n$ approximates this series with $|\text{error}| < |a_{n+1}| = \frac{1}{(n+1)^5}$. To force $\frac{1}{(n+1)^5} < .001$ we need $(n+1)^5 > 1000$. $n = 3$ works. Therefore

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^5} \approx s_3 = \frac{1}{1^5} - \frac{1}{2^5} + \frac{1}{3^5} = 1 - \frac{1}{32} + \frac{1}{243} \approx 0.973.$$
2. Find the radius of convergence and the interval of convergence of the power series:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^n}{n} (x - 1)^n.$$ 

**Solution:** Applying the Ratio Test to the series $$\sum_{n=1}^{\infty} \frac{2^n}{n} |x - 1|^n$$ we obtain

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{2^{n+1}|x - 1|^{n+1}(n + 1)}{2^n|x - 1|^n n} = 2|x - 1|.$$ 

Therefore the radius of convergence is $$R = \frac{1}{2}$$ and we get absolute convergence when $$|x - 1| < \frac{1}{2}$$ and divergence when $$|x - 1| > \frac{1}{2}$$. So far, we have convergence in the interval $$(0.5, 1.5)$$. We will now check the endpoint behavior. Evaluating at $$x = 0.5$$ we obtain

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^n}{n} (-0.5)^n = -\sum_{n=1}^{\infty} \frac{1}{n},$$

which diverges. Evaluating at $$x = 1.5$$ we obtain

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^n}{n} (0.5)^n = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n},$$

which converges as an alternating series. So the interval of convergence is $$(0.5, 1.5]$$. 

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3. Find the Taylor series associated with \( f(x) = (1 + 2x)^{\frac{1}{2}} \) expanded about \( a = 0 \). You can either express the answer in summation notation or describe the pattern somehow (if you do, compute at least 6 terms of the series).

**Solution:** The derivatives of \( f(x) \) are

\[
(1+2x)^{\frac{1}{2}}, \quad (1+2x)^{-\frac{3}{2}}, \quad -(1+2x)^{-\frac{3}{2}}, \quad 3(1+2x)^{-\frac{3}{2}}, \quad -3\cdot5(1+2x)^{-\frac{3}{2}}, \quad 3\cdot5\cdot7(1+2x)^{-\frac{3}{2}}, \quad \ldots.
\]

Evaluating at \( a = 0 \) we obtain

\[
1, \quad 1, \quad -1, \quad 3, \quad -3 \cdot 5, \quad 3 \cdot 5 \cdot 7, \quad \ldots.
\]

Therefore the Taylor Series is

\[
\frac{1}{0!} + \frac{1}{1!}x - \frac{1}{2!}x^2 + \frac{3}{3!}x^3 - \frac{3 \cdot 5}{4!}x^4 + \frac{3 \cdot 5 \cdot 7}{5!}x^5 - \ldots =
\]

\[
1 + x + \sum_{n=2}^{\infty} (-1)^{n-1}\frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{n!}x^n.
\]
4. In the following problem, Part (a) and Part (b) refer to \( f(x) = (1 + 2x)^{\frac{1}{2}} \) and \( a = 0 \).

(a) Find \( P_5(x; 0) \), the 5th-order Taylor polynomial for \( f(x) \) expanded about \( a = 0 \). (You’ve done most of the work in Problem 3 already.)

(b) Using the error bound \( \frac{M_{n+1}|x-a|^{n+1}}{(n+1)!} \), compute an upper bound to the error in using \( P_5(2; 0) \) to approximate the value of \( f(2) \).

**Solution:**

\[
P_5(x; 0) = 1 + \frac{1}{11}x - \frac{1}{22}x^2 + \frac{3}{33}x^3 - \frac{35}{44}x^4 + \frac{357}{55}x^5.
\]

We have

\[
f^{(6)}(x) = -3 \cdot 5 \cdot 7 \cdot 9 (1 + 2x)^{-\frac{11}{2}},
\]

and \( M_6 \) is the largest absolute value of this expression using \( x \in [0, 2] \), that is \( M_6 = 3 \cdot 5 \cdot 7 \cdot 9 \). Therefore an upper bound to the error is

\[
\frac{M_6|2 - 0|^{6}}{6!} = \frac{3 \cdot 5 \cdot 7 \cdot 8 \cdot 64}{720} = 84.
\]
5. (a) Using the results of Problem 3, express \( \int_{0}^{0.1} (1 + 2x)^{\frac{1}{2}} \, dx \) as an infinite series.

(b) Using the Alternating Series Estimation Theorem, find \( n \) so that \( s \approx s_n \) with \(|\text{error}| < 0.01\), where \( s \) is the alternating series found in Part (a). Note that apart from the first couple of terms, the answer in Part (a) is an alternating series.

**Solution:** Technically we should prove that the Taylor series for \( (1 + 2x)^{\frac{1}{2}} \) converges to \( (1 + 2x)^{\frac{1}{2}} \) for \( 0 \leq x \leq 0.1 \), but it does since

\[
\frac{M_{n+1}|0.1 - 0|^{n+1}}{(n+1)!} = \frac{1 \cdot 3 \cdot \cdots \cdot 2n - 1}{(n+1)!} (0.1)^{n} \leq 2^n (0.1)^n = (0.2)^n \to 0.
\]

But I did not ask you to prove this. So now we have

\[
\int_{0}^{0.1} (1 + 2x)^{\frac{1}{2}} \, dx = \int_{0}^{0.1} \left( \frac{1}{0!} x + \frac{1}{1!} x^2 - \frac{1}{2!} x^3 + \frac{3}{3!} x^4 - \frac{3 \cdot 5}{4!} x^5 + \frac{3 \cdot 5 \cdot 7}{5!} x^6 - \cdots \right) \, dx = \left[ \frac{1}{0!} (0.1) + \frac{1}{1!} \frac{(0.1)^2}{2} - \frac{1}{2!} \frac{(0.1)^3}{3} + \frac{3}{3!} \frac{(0.1)^4}{4} - \frac{3 \cdot 5}{4!} \frac{(0.1)^5}{5} + \frac{3 \cdot 5 \cdot 7}{5!} \frac{(0.1)^6}{6} - \cdots \right]_{0}^{0.1} = \frac{1}{0!} (0.1) + \frac{1}{1!} \frac{(0.1)^2}{2} - \frac{1}{2!} \frac{(0.1)^3}{3} + \frac{3}{3!} \frac{(0.1)^4}{4} - \frac{3 \cdot 5}{4!} \frac{(0.1)^5}{5} + \frac{3 \cdot 5 \cdot 7}{5!} \frac{(0.1)^6}{6} - \cdots
\]

which looks like an alternating series if we group together the first two terms and set

\[
a_1 = \frac{1}{0!} (0.1) + \frac{1}{1!} \frac{(0.1)^2}{2},
\]

\[
a_2 = -\frac{1}{2!} \frac{(0.1)^3}{3},
\]

\[
a_3 = \frac{3}{3!} \frac{(0.1)^4}{4},
\]

etc. The error in using \( s_1 = \frac{1}{0!} (0.1) + \frac{1}{1!} \frac{(0.1)^2}{2} \) is no greater than

\[
|a_2| = \frac{1}{2!} \frac{(0.1)^3}{3} = 0.0001667,
\]

so we can say

\[
\int_{0}^{0.1} (1 + 2x)^{\frac{1}{2}} \, dx \approx \frac{1}{0!} (0.1) + \frac{1}{1!} \frac{(0.1)^2}{2} = 0.105
\]
with error < 0.01.

Check:

$$\int_0^{0.1} (1 + 2x)^{\frac{1}{2}} \, dx = \frac{1}{3} (1 + 2x)^{\frac{3}{2}} \Bigg|_0^{0.1} = \frac{1}{3} \left( (1.2)^{\frac{3}{2}} - 1 \right) = 0.104845.$$  

The error is 0.000155.