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1. Evaluate the integral $\int_0^\pi x^2 \sin x \, dx$ using integration by parts.

Solution: Setting $u = x^2$, $dv = \sin x \, dx$, $du = 2x \, dx$, $v = -\cos x$, we have

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2 \int x \cos x \, dx.$$  

Setting $u = x$, $dv = \cos x \, dx$, $du = dx$, $v = \sin x$, we have

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x.$$  

Putting everything together we get

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x.$$  

Using the limits of integration, the answer is $\pi - 2$.

2. Evaluate the trigonometric integral $\int_0^\pi \sec^4 x \tan^3 x \, dx$. Do not use the integral tables.

Solution: Method 1:

$\sec^4 x \tan^3 x \, dx = (1 + \tan^2 x) \tan^3 x \cdot \sec^2 x \, dx = (1 + u^2)u^3 \, du$

where $u = \tan x$.

Method 2:

$\sec^4 x \tan^3 x \, dx = \sec^3 x(\sec^2 x - 1) \cdot \sec x \tan x \, dx = u^3(u^2 - 1) \, du$

where $u = \sec x$.

Method 3:

$\sec^4 x \tan^3 x = (1 + \tan^2 x)^2 \tan^3 x = \tan^3 x + 2 \tan^5 x + \tan^7 x$.

By any method the answer is $\frac{5}{12}$. 

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3. Evaluate the integral \( \int_0^1 \frac{1}{(1+x^2)^{5/2}} \, dx \) using a trigonometric substitution. Do not use the integral tables.

**Solution:** Setting \( x = \tan \theta \), \( dx = \sec^2 \theta \, d\theta \), we have

\[
\int_0^1 \frac{1}{(1+x^2)^{5/2}} \, dx = \int_0^{\frac{\pi}{4}} \frac{\sec^2 \theta}{\sec^5 \theta} \, d\theta = \int_0^{\frac{\pi}{4}} \cos^3 \theta \, d\theta = \int_0^{\frac{\pi}{4}} (1 - \sin^2 \theta) \cos \theta \, d\theta.
\]

Setting \( u = \sin \theta \), \( du = \cos \theta \, d\theta \), we have

\[
\int_0^{\frac{\pi}{4}} (1 - \sin^2 \theta) \cos \theta \, d\theta = \int_0^{\frac{\pi}{4}} 1 - u^2 \, du = \frac{5}{12} \sqrt{2}.
\]

4. Evaluate the integral \( \int_3^5 \frac{x^3}{x^2-3x+2} \, dx \) using partial fraction decomposition. Do not use the integral tables.

**Solution:** Long division yields

\[
\frac{x^3}{x^2-3x+2} = x + 3 + \frac{7x-6}{x^2+3x+2}.
\]

Partial fraction decomposition yields

\[
\frac{7x-6}{x^2+3x+2} = \frac{8}{x-2} - \frac{1}{x-1}.
\]

Therefore

\[
\int \frac{x^3}{x^2-3x+2} \, dx = \int x + 3 + \frac{8}{x-2} - \frac{1}{x-1} \, dx = \frac{x^2}{2} + 3x + 8 \ln |x-2| - \ln |x-1| + C.
\]

Now use the limits of integration.

5. (a) Approximate the integral \( \int_1^2 \frac{x^5}{1+x^4} \, dx \) using the Trapezoidal Rule and \( n = 4 \) intervals.

(b) Using the error bound \( |E_T| \leq \frac{K(b-a)^3}{12n^2} \), how large does \( n \) have to be so that the error in the trapezoidal approximation to \( \int_1^2 \frac{x^5}{1+x^4} \, dx \) is less than \( 10^{-6} \)?

**Solution:** We have \( f(x) = \frac{x^5}{1+x^4} \) and \( \Delta = \frac{2-1}{4} = .125 \). Therefore

\[
T_4 = .125(f(1.00) + 2f(1.25) + 2f(1.50) + 2f(1.75) + f(2.00)) = 1.84445.
\]
With some effort we find that
\[ f''(x) = \frac{20x^3 + 24x^6 + 6x^9 + 2x^{12}}{(1 + x^3)^4}. \]

Plotting this between 1 and 2 we find that we can use \( K = 4 \) as a rough upper bound. Solving for \( n \) in \( \frac{K(b-a)^3}{12n^4} \leq 10^{-6} \) yields \( n \geq 578 \).