A total of eight problems will be graded. Indicate the problems that you wish to have graded by circling the problem number on the exam. All work must be contained in your exam booklet. Please turn in your exam and exam booklet when you are finished with the exam.

1. Let $A$ be an $n \times n$ complex matrix such that $x$ is an eigenvector of $A$ corresponding to eigenvalue 0. Let $y$ be an element of $C^n$ satisfying $A^2y = x$ such that $Ay$ is not an eigenvector of $A$ corresponding to eigenvalue 0. Show that $\{y, Ay, A^2y\}$ is a linearly independent subset of $C^n$.

2. Let $M_{m \times n}$ denote the vector space of $m \times n$ matrices over the complex numbers $C$. Let $A$ be a fixed element of $M_{p \times n}$, and let $L : M_{n \times q} \to M_{p \times q}$ be defined by $L(X) = AX$.
   (a) Show that $L$ is a linear mapping.
   (b) Show that not all linear mappings from $M_{n \times q}$ to $M_{p \times q}$ are of the form $L(X) = AX$ for some fixed matrix $A$ in $M_{p \times n}$.

3. Let $L$ be a linear mapping from a vector space $U$ to a vector space $V$. Assume, for some $x_0$ and $y_0$, that $L(x_0) = y_0$. Show that the solution set to $L(x) = y_0$ in $U$ is $\{x_0\} + \ker(L)$, where $\ker(L)$ denotes the kernel of $L$.

4. Assume that the probability distribution of $X$ is the negative binomial distribution with discrete pdf given by $f(x; r, p) = \binom{x-1}{r-1} p^r q^{x-r}; x = 0, 1, 2, \ldots$. Find $E(X^2)$.
   (Show your work.)

5. Consider independent random samples from two independent exponential distributions, $X_i \sim EXP(\theta_1)$ and $Y_j \sim EXP(\theta_2); i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$. Show that
   $$\frac{\theta_2 \bar{X}}{\theta_1 \bar{Y}} \sim F_{2n_1, 2n_2}.$$  
   (Do not need to prove but just argue using standard results.)

6. We know that $\bar{X}$ is approximately $N(\mu, \sigma^2/n)$ for large $n$. Find the approximate distribution of $u(\bar{X}) = \bar{X}^3$. Hint: Apply the result that $g(\bar{X})$ is approximately
   $$N(g(\mu), Var(\bar{X}) \times (g'(\mu))^2).$$

7. Let $x_1 = 1$ and $x_{n+1} = \frac{x_{n+1}}{4}$ for $n \geq 2$. Show that the sequence $(x_n : n \geq 1)$ is decreasing and bounded. What is its limit?
8. Let \( G = \{(x, y) \in \mathbb{R}^2 : 4x^2 + 9y^2 = 36 \} \). Show that \( G \) is a closed subset of \( \mathbb{R}^2 \).

9. Show that there exists \( 0 < p < 1 \) such that \( \cos p = p \).

10. Let \( f \) be a smooth function. Using Taylor’s Theorem, find the coefficients \( A, B, C \) such that

\[
f'(x) = Af(x) + Bf(x + h) + Cf(x + 2h) + O(h^2).
\]

11. Let \( f \in C^2(D) \) for a given domain \( D \subset \mathbb{R} \), with \( f(\alpha) = 0 \) for some \( \alpha \in D \). For a given \( x_n \in D \), define

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.
\]

Assuming the sequence \( x_n \) converges to \( \alpha \) and \( f'(x) \neq 0, f''(x) \neq 0 \) for every \( x \in D \), find the order of convergence and the asymptotic constant. That is, find the constants \( m \) and \( \lambda \neq 0 \) such that

\[
\lim_{n \to \infty} \frac{|\alpha - x_{n+1}|}{|\alpha - x_n|^m} = \lambda.
\]

12. Find the quadratic polynomial that interpolates to \( y = \sqrt{x} \) at the nodes \( x_0 = 1/4 \), \( x_1 = 9/16 \), and \( x_2 = 1 \). What is the upper bound on the error over the interval \([1/4,1]\)?