Mathematics Comprehensive Exam
March 17, 2004

A total of eight problems will be graded, including at least one problem from each of the six areas below. Indicate the problems that you wish to have graded by circling the problem number on the exam. All work must be contained in your exam booklet. Please turn in your exam and exam booklet when you are finished with the exam.

Linear Algebra
1. Let $M_{m \times n}$ denote the vector space of $m \times n$ matrices over the complex numbers $C$. Let $A$ be a fixed element of $M_{p \times n}$, and let $L : M_{n \times q} \rightarrow M_{p \times q}$ be defined by $L(X) = AX$.
   (a) Show that $L$ is a linear mapping.
   (b) Show that not all linear mappings from $M_{n \times q}$ to $M_{p \times q}$ are of the form $L(X) = AX$ for some fixed matrix $A$ in $M_{p \times n}$.

2. Let $L$ be a linear mapping from a vector space $U$ to a vector space $V$. Assume, for some $x_0$ and $y_0$, that $L(x_0) = y_0$. Show that the solution set to $L(x) = y_0$ in $U$ is $\{x_0\} + \ker(L)$, where $\ker(L)$ denotes the kernel of $L$.

Statistics
3. Assume that the probability distribution of $X$ is the negative binomial distribution with discrete pdf given by $f(x; r, p) = \binom{x-1}{r-1} p^r q^{x-r}; x = 0, 1, 2, \ldots$. Find $E(X^2)$.
   (Show your work.)

4. Consider independent random samples from two independent exponential distributions, $X_i \sim \text{EXP}(\theta_1)$ and $Y_j \sim \text{EXP}(\theta_2); i = 1, 2, \ldots, n_1, j = 1, 2, \ldots, n_2$. Show that $\frac{\theta_2}{\theta_1} \frac{\overline{X}}{\overline{Y}} \sim F_{2n_1, 2n_2}$.
   (Do not need to prove but just argue using standard results.)

Real Analysis
5. Let $x_1 = 1$ and $x_{n+1} = \frac{x_n + 1}{4}$ for $n \geq 2$. Show that the sequence $(x_n : n \geq 1)$ is decreasing and bounded. What is its limit?

6. Show that there exists $0 < p < 1$ such that $\cos p = p$. 

Numerical Analysis

7. Let \( f \in C^2(D) \) for a given domain \( D \subset \mathbb{R} \), with \( f(\alpha) = 0 \) for some \( \alpha \in D \). For a given \( x_n \in D \), define

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.
\]

Assuming the sequence \( x_n \) converges to \( \alpha \) and \( f'(x) \neq 0, f''(x) \neq 0 \) for every \( x \in D \), find the order of convergence and the asymptotic constant. That is, find the constants \( m \) and \( \lambda \neq 0 \) such that

\[
\lim_{n \to \infty} \left| \frac{\alpha - x_{n+1}}{\alpha - x_n} \right|^m = \lambda.
\]

8. Find the quadratic polynomial that interpolates to \( y = \sqrt{x} \) at the nodes \( x_0 = 1/4, x_1 = 9/16, \) and \( x_2 = 1 \). What is the upper bound on the error over the interval \([1/4,1]\)?

Computer Organization

9. Design a pipeline (by listing the various stages and their order) that will work well for executing the integer instructions of an instruction set architecture that has at most one memory-based operand for each of the ALU operations. Identify the hazards present in your pipeline. For each hazard for which at least one stall cycle remains after bypassing etc., indicate how many stall cycles are required. You may assume that all memory accesses hit the cache and that cache hits are accomplished in one cycle.

10. second problem

Operating Systems

11. Consider a system with a total of 145 units of memory, allocated to three processes as shown:

<table>
<thead>
<tr>
<th>Process</th>
<th>Max Required</th>
<th>Allocated</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>70</td>
<td>45</td>
</tr>
<tr>
<td>2</td>
<td>60</td>
<td>40</td>
</tr>
<tr>
<td>3</td>
<td>60</td>
<td>15</td>
</tr>
</tbody>
</table>

Using the banker’s algorithm, determine if it is safe to allow a fourth process to start. This fourth process requires 25 memory units initially, with a maximum of memory requirement of 60. Justify your answer.

12. second problem