A total of eight problems will be graded. Indicate the problems that you wish to have graded by circling the problem number on the exam. All work must be contained in your exam booklet. Please turn in your exam and exam booklet when you are finished with the exam.

**Linear Algebra Problems:**
1. Assume \( n \geq 2 \), \( b \) is a nonzero column vector in \( \mathbb{R}^n \), and \( B = bb^T \).
   a. Show \( b \) is an eigenvector of \( B \) and find its corresponding eigenvalue.
   b. Show 0 is an eigenvalue of \( B \).
2. Show that if \( U \) is a vector space of dimension \( n > 0 \) and \( S = \{s_1, \ldots, s_n\} \) is a subset of \( U \) such that \( \text{span}(S) = U \), then \( S \) is linearly independent.
3. Let \( U \) and \( V \) be vector spaces over \( K \), and let \( L : U \to V \) be a linear mapping. If \( U \) is finite dimensional and \( \dim(U) = n > 0 \), show how to find a basis for the range of \( L \). Prove your answer.

**Analysis Problems:**
4. Let \( G = \{(x, y) : 4x^2 + 9y^2 = 36\} \). Show that \( G \) is a closed subset of \( \mathbb{R}^2 \).
5. Show that there exists \( 0 < p < 1 \) such that \( \cos p = p \).
6. Show that the function \( f(x) = \frac{1}{x} \) is uniformly continuous on the interval \([1/4, \infty)\).

**Probability and Statistics Problems:**
7. Let \( E(Y|X = x) = 2x \), \( \text{Var}(Y|X = x) = 4x^2 \), and let \( X \) have a uniform distribution on the interval \((0, 1)\). What is \( \text{Var}(Y) \)?
   a. \( \frac{1}{3} \)
   b. \( \frac{4}{3} \)
   c. \( \frac{3}{4} \)
   d. \( \frac{7}{4} \)
   e. cannot be determined from the information given.
8. Let \( X_1 \) and \( X_2 \) be independent continuous random variables, each with density function
   \[
   f(x) = \begin{cases} 
   \lambda e^{-\lambda x} & \text{for } x > 0 \\
   0 & \text{otherwise},
   \end{cases}
   \]
where $\lambda > 0$. Let $Y_1 = X_1 + 2X_2$ and $Y_2 = 2X_1 + X_2$. What is the joint density function $g(y_1, y_2)$ for $y_1 > 0$ and $y_2 > 0$?

a. $\frac{1}{3} \lambda^2 e^{-\lambda(y_1+y_2)/3}$

b. $\lambda^2 e^{-\lambda(y_1+y_2)/3}$

c. $3\lambda^2 e^{-\lambda(y_1+y_2)/3}$

d. $3\lambda^2 e^{-3\lambda(y_1+y_2)}$

e. $\frac{1}{5} \lambda^2 e^{-\lambda(y_1+y_2)/3}$

9. Let $Y$ have a uniform distribution on the interval $(0, 1)$, and let the conditional distribution of $X$ given $y$ be uniform on the interval $(0, \sqrt{y})$. What is the marginal density function of $X$, for $0 < x < 1$?

a. $2(1 - x)$

b. $2x$

c. $2(1 - x^{\frac{1}{2}})$

d. $\frac{1}{\sqrt{x}} - 1$

e. $\frac{1}{2\sqrt{x}}$

**Numerical Analysis Problems:**

10. By constructing a divided difference table at the points $\{x_k : k = 0, 1, 2, 3\} = \{0, 1, 2, 4\}$, find the cubic interpolation polynomial for $f(x) = \sqrt{x}$ for $x \in [0, 4]$.

11. Consider a fixed point iteration

\[ x_{n+1} = g(x_n) \]

where $g \in C^p$ and $\alpha = g(\alpha)$. Recall that if $g'(\alpha) = g''(\alpha) = \cdots = g^{(p-1)}(\alpha) = 0$ but $g^{(p)}(\alpha) \neq 0$, then the fixed point iteration converges with order $p$ for $x_0$ sufficiently close to $\alpha$. Prove that the Newton’s method has at least order of convergence 2.

12. Apply Simpson’s rule with $f(x) = \sqrt{1+x^3}$ and $h = \frac{1}{4}$ to approximate the integral

\[ I = \int_0^1 f(x) \, dx \]

Also estimate the size of the error with $|f^{(4)}(x)| < 8$ for any $x \in [0, 1]$. 