Department of Mathematics and Statistics  
Master’s Comprehensive Exam  

Date: October 16, 2009, Friday  
Time: 9:00am to 12:00pm  

Directions: A total of eight problems will be graded. Indicate the problems that you wish to have graded by circling the problem number on the exam. All work must be contained in your exam booklet. Please turn in your exam and exam booklet when you are finished with the exam.

**Linear Algebra Problems:**

1: Let $U$ and $W$ be vector spaces over $\mathbb{R}$. Then $U \times W = \{(u, w) \mid u \in U \text{ and } w \in W\}$ is a vector space with vector addition $(u_1, w_1) + (u_2, w_2) = (u_1 + u_2, w_1 + w_2)$ and scalar multiplication $c \cdot (u_1, w_1) = (cu_1, cw_1)$ for all $u_1, u_2 \in U$, $w_1, w_2 \in W$ and all scalars $c$. Suppose that $U$ and $W$ are finite dimensional with $\dim U = m$ and $\dim W = n$. Show that $\dim U \times W = m + n$.

2: Let $U$ and $W$ be subspaces of a real vector space $V$. Then $U + W = \{u + w \mid u \in U \text{ and } w \in W\}$ is also a subspace of $V$. Show that 

$$\dim U + \dim W = \dim(U + W) + \dim(U \cap W).$$

(Hint. Show that the map 

$$L : U \times W \to V$$

given by 

$$L(u, w) = u - w$$

is a linear map. What is its image? What is its kernel?)

3: Let $A$ be an $m \times n$ real matrix. Show that the rank of $A^T A$ is equal to the rank of $A$. 

1
Real Analysis Problems:

4: Find a real sequence \((x_n)\) such that \(\lim_{n \to \infty} (x_{n+1} - x_n) = 0\), but \((x_n)\) is not convergent. Explain why it is not convergent.

5: A number \(s\) is a subsequential limit of a sequence \((x_n)\) if \((x_n)\) has a subsequence that is convergent to \(s\).
   
a. Find a real sequence and a subsequential limit of \((x_n)\) that is not an accumulation point of the range of \((x_n)\).
   
b. Find a real sequence \((x_n)\) such that every subsequential limit of \((x_n)\) is an accumulation point of the range of \((x_n)\).

6: Show that if a real sequence \((x_n)\) is bounded below by \(a\) and above by \(b\), then all subsequential limits of \((x_n)\) belong to \([a, b]\).

Numerical Analysis Problems:

7: Construct a Taylor polynomial for \(f(x) = x \sin(2x)\), that is accurate to within \(10^{-5}\) maximum error over the interval \([-1/2, 1/2]\).

8: Consider the fixed-point iteration \(x_{n+1} = 1 + e^{-x_n}\). Show that this iteration converges for any \(x_0 \in [1, 2]\). How many iterations will take to achieve \(10^{-5}\) accuracy?

9: Let \(I(f) = \int_0^1 e^{-x^2} dx\) and \(T_n(f)\) be the \(n\)-subinterval trapezoid rule approximation of \(I(f)\), with uniform grid size \(h\). How small does \(h\) have to be to guarantee that the error is less than \(10^{-5}\).
Theory of Statistics I Problems:

Show all your work.
Select the correct response for each problem worked.
No work means no credit.

10: Let $X$ be a continuous random variable with joint density function

$$ f(x, y) = \begin{cases} 
  xy & \text{for } 0 \leq x \leq 2 \text{ and } 0 \leq y \leq 1 \\
  0 & \text{otherwise.}
\end{cases} $$

What is the $P[\frac{X}{2} \leq Y \leq X]$?

(a) $\frac{3}{32}$
(b) $\frac{1}{8}$
(c) $\frac{1}{4}$
(d) $\frac{3}{8}$
(e) $\frac{3}{4}$

11: A box contains 4 red balls and 6 white balls. A sample of size 3 is drawn without replacement from the box. What is the probability of obtaining 1 red and 2 white balls given that at least 2 of the balls in the sample are white?

(a) $\frac{1}{2}$
(b) $\frac{2}{3}$
(c) $\frac{3}{4}$
(d) $\frac{9}{11}$
(e) $\frac{54}{35}$

12: Let $X$ and $Y$ be continuous random variables with joint density function

$$ f(x, y) = \begin{cases} 
  2(x + y) & \text{for } 0 < x < y < 1 \\
  0 & \text{otherwise.}
\end{cases} $$

Then $E[Y] =$

(a) $\frac{5}{12}$
(b) $\frac{1}{2}$
(c) $\frac{3}{4}$
(d) 1
(e) $\frac{7}{6}$
Attempt questions either from Theory of Statistics II or Regression Analysis:

Theory of Statistics II Problems: Show your work for full credit

13: Let $X_1, X_2, \ldots X_n$ be a random sample from a discrete distribution with p.m.f. given by

$$f(x; \theta) = \begin{cases} \frac{\theta^x - a}{x!} & \text{for } x = 0, 1, 2, \ldots \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Find the m.l.e. of the parameter $\theta$.

(b) Prove that $\bar{X}$ is the efficient estimator of the parameter $\theta$.

14: (a) Let $X_1, X_2, \ldots X_n$ be a random sample from a distribution with parameter $\theta$. Define a sufficient statistic for a parameter $\theta$.

(b) Find the sufficient statistic of the parameter $\theta$ from the distribution

$$f(x; \theta) = \begin{cases} \theta x^{\theta - 1} & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

15: Let $X_1, X_2, \ldots X_n$ be a random sample from a distribution

$$f(x; \theta) = \begin{cases} \frac{1}{\theta^4} x^3 e^{-x/\theta} & \text{for } 0 < x < \infty, \ 0 < \theta < \infty \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Find the complete sufficient statistic for $\theta$.

(b) Find the Minimum Variance Unbiased Estimator (MVUE) for $\theta$. 
Regression Analysis Problems: Show your work for full credit

13: Consider the regression model \( y = \beta_0 + \beta_1 x + \varepsilon \), with \( E(\varepsilon) = 0 \), \( V(\varepsilon) = \sigma^2 \) and \( \varepsilon \) uncorrelated. Show that \( \text{Cov}(\hat{\beta}_0, \hat{\beta}_1) = -\bar{x}\sigma^2 / S_{xx} \) where \( S_{xx} = \sum_{i=1}^{n}(x_i - \bar{x})^2 \)

14: Consider the Generalized Least Squares Regression Model

\[ Y = X\beta + \varepsilon \] with \( E(\varepsilon) = 0 \) and \( V(\varepsilon) = \sigma^2 V \)

where \( V \) is the variance-covariance matrix with unequal diagonal elements and zero off diagonal elements.

(a) Prove in this case, \( \hat{\beta} = (X'V^{-1}X)^{-1}X'V^{-1}Y \)

(b) Find \( E(\hat{\beta}) \) and \( V(\hat{\beta}) \).

15: (a) What do you understand by the model adequacy checking in regression analysis? How do you test the model adequacy?

(b) Explain the following terms in the regression analysis

* Residuals

* Standardized Residuals

* Studentized Residuals

* PRESS Residuals