Choose two of the following eight:

(1) Identify all sequences in $\Sigma_2$ which are periodic points of period 3 for $\sigma$. Which points lie on the same orbit? Repeat for periods 4 and 5.

(2) Let $\Sigma'$ consist of all sequences in $\Sigma_2$ which never have two consecutive zeros (i.e. 0 is always followed by a 1).
   (a) Show that $\Sigma'$ is a closed, invariant subset of $\Sigma_2$ under $\sigma$.
   (b) Show that periodic points are dense in $\Sigma'$.
   (c) Show that there is a dense orbit in $\Sigma'$.
   (d) How many fixed points are there for $\sigma, \sigma^2$ and $\sigma^3$ in $\Sigma'$?
   (e) (Optional) Find a recursive formula for the number of fixed points of $\sigma^n$ in terms of the number of fixed points of $\sigma^{n-1}$ and $\sigma^{n-2}$.

(3) Let $s \in \Sigma_2$. Define the stable set of $s$, $W_s(s)$, to be the set of sequences $t$ such that $d(\sigma^i(t), \sigma^i(s)) \to 0$ as $i \to \infty$. Identify all of the sequences in $W_s(s)$.

(4) Consider the tent map on $[0, 1]$: 
   $$T_2(x) = \begin{cases} 
   2x & 0 \leq x \leq \frac{1}{2} \\
   2 - 2x & \frac{1}{2} \leq x \leq 1 
   \end{cases}$$
   We would like to establish the symbolic dynamics of $T_2$, but first we must modify $\Sigma_2$. Notice there is an ambiguity at $x = \frac{1}{2}$ being both in $I_0$ and $I_1$. In fact, the itineraries of any $p/2^k$ will encounter this ambiguity. We therefore modify the sequence space $\Sigma_2$ so that $(s_0, s_1, ..., s_{k-1}c10)$ are identified for $c = 0$ or 1. Denote $\Sigma_2$ with this identification by $\tilde{\Sigma}$.
   (a) Prove that $S : I \to \tilde{\Sigma}$ is one-to-one, where $S(x)$ is the itinerary of $x$.
   (b) Prove that $\sigma \circ S = S \circ T_2$. (Since $S(x)$ was shown in class to be onto and continuous with continuous inverse, notice that this establishes a conjugacy with $(\tilde{\Sigma}, \sigma)$ and hence $T_2$ is chaotic.)
   (c) Given the above, is $T_2$ topologically conjugate to the quadratic map $f_\lambda(x) = 4x(1-x)$? Why or why not?

(5) A point $p$ is called non-wandering for $f$, if, for any open interval $J$ containing $p$, there exists $x \in J$ and $n > 0$ such that $f^n(x) \in J$. (Note that $p$ itself may not return to $J$.) Let $\Omega(f)$ denote the set of non-wandering points for $f$.
   (a) Prove that $\Omega(f)$ is closed.
   (b) If $f_\lambda$ is the quadratic map with $\lambda > 2 + \sqrt{5}$, show that $\Omega(f_\lambda) = \Lambda$, where $\Lambda$ is the invariant Cantor set of $f_\lambda$.
   (c) Identify $\Omega(f_\lambda)$ for each $0 < \lambda \leq 3$.

(6) A point $p$ is called recurrent for $f$ if, for any open interval $J$ containing $p$, there exists $n > 0$ such that $f^n(p) \in J$. Clearly all periodic points are recurrent.
   (a) Give an example of a non-periodic recurrent point for $f_\lambda$ with $\lambda > 2 + \sqrt{5}$.
   (b) Give an example of a non-wandering point for $f_\lambda$ which is not recurrent.